Continuous Random Variables

Statistics I · Section 6

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Outline

6.1 The uniform distribution
6.2 The exponential distribution
6.3 The Gamma distribution
6.4 The normal distribution

From a pmf to a pdf

A continuous variable X has a density f. Three facts from Section 4:

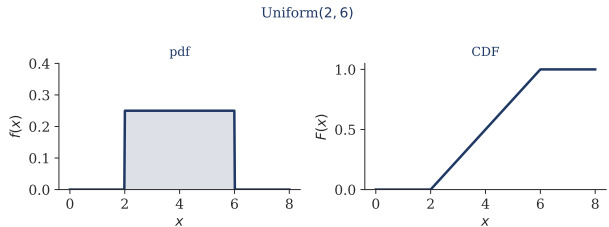
- A probability is an area under f: $P(a < X \leq b) = \int_a^b f(x)\,dx$.
- A single point has probability zero: $P(X = x) = 0$. So $<$ and $\leq$ give the same probability.
- The CDF and the density are linked by $F'(x) = f(x)$ wherever f is continuous.

The Uniform Distribution

The uniform distribution

Definition 6.1 (Uniform)

$X \sim \mathrm{U}(a, b)$ (均匀分配) has the constant density $f(x) = \frac{1}{b-a}$ on $[a, b]$, $a < b$.



- A uniform variable is equally likely anywhere in $[a, b]$. Probability is proportional to length.

Uniform CDF

Proposition 6.2

$F(x) = \dfrac{x-a}{b-a}$ on $[a, b]$ (and 0 below a, 1 above b).

- Integrate the constant density from a. The result is a straight ramp.

Uniform mean and variance

Proposition 6.3

$$\mathbb{E}(X) = \frac{a+b}{2}, \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}.$$

- By direct integration. That is cleaner here than pulling the moments out of the MGF.

Example 6.4

Example 6.4

$X \sim \mathrm{U}(0, 10)$. Find $P(X \leq 3)$, $P(2 < X < 7)$, $\mathbb{E}(X)$, $\mathrm{Var}(X)$.

Example 6.5

Example 6.5

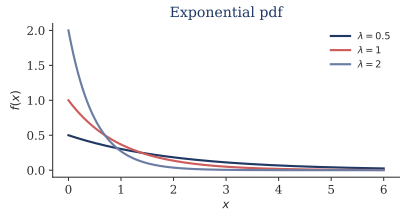
A bus is equally likely to arrive anytime in the next 15 min. Find $P(\text{wait} > 10)$ and the expected wait.

The Exponential Distribution

The exponential distribution

Definition 6.6 (Exponential)

$X \sim \text{Exp}(\lambda)$ (指數分配) has density $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$.



- It is the model for a waiting time. The geometric played this role in the discrete case.

Exponential CDF and tail

Proposition 6.7

$F(x) = 1 - e^{-\lambda x}$, $P(X > x) = e^{-\lambda x}$ $(x \geq 0)$.

- Integrate the density. The tail has a simple form, and Theorem 6.11 uses it.

The wait in a Poisson process

Proposition 6.8

Suppose the number of events in any window of length x is Poisson(λx). Let X be the wait until the first event. Then $P(X > x) = e^{-\lambda x}$, so $X \sim \text{Exp}(\lambda)$.

- The wait is longer than x exactly when no event arrives in $[0, x]$.
- The Poisson counts the events. The exponential measures the time between them.

Exponential MGF, mean, variance

Proposition 6.9

$M_X(t) = \dfrac{\lambda}{\lambda - t}$ $(t < \lambda)$; $\mathbb{E}(X) = 1/\lambda$, $\mathrm{Var}(X) = 1/\lambda^2$.

- Integrate $\int_0^\infty e^{tx}\lambda e^{-\lambda x}dx$, then differentiate at 0.
- The rate and the mean are reciprocals. A mean wait of 4 hours means $\lambda = 0.25$ per hour.

Example 6.10

Example 6.10

Events occur at rate $\lambda = 0.5$/hour. Find $P(X > 3)$, $\mathbb{E}(X)$, $\mathrm{Var}(X)$, $P(X < 1)$.

What the past wait tells us

Theorem 6.11

For $s, t \geq 0$, $P(X > s + t \mid X > s) = P(X > t)$.

- The exponential is memoryless, since $e^{-\lambda(s+t)}/e^{-\lambda s} = e^{-\lambda t}$.
- A used component is "as good as new." The geometric is the only discrete distribution that behaves this way.

Example 6.12

Example 6.12

A component's lifetime is exponential with mean 1000 h. Find $P(X > 1500)$ and, by memorylessness, $P(X > 1500 \mid X > 500)$.

The geometric and the exponential

The exponential is the continuous version of the geometric.

	Geometric	**Exponential**
X measures	trials until the first success	time until the first event
Support	$1, 2, 3, \ldots$	every $x \geq 0$
$P(X > x)$	q^x, with $q = 1 - p$	$e^{-\lambda x}$
Mean	$1/p$	$1/\lambda$
$P(X > s + t \mid X > s)$	$P(X > t)$	$P(X > t)$

Example 6.13

Example 6.13

Trades in a stock arrive as a Poisson process at 4 per minute. Let X be the wait in minutes until the next trade.

(a) Find $P(X > 0.5)$ in two ways: from the exponential tail, and from the number of trades in half a minute.

(b) Find the mean wait in seconds.

(c) Find the probability that the wait is under 15 seconds.

From a uniform to any distribution

- **Remark (probability integral transform).** If X has a continuous, strictly increasing CDF F, then $F(X)$ is uniform on $(0, 1)$. If U is uniform on $(0, 1)$, then $F^{-1}(U)$ has CDF F.
- We state this without proof.
- A computer draws U and returns $F^{-1}(U)$. For the exponential, $F^{-1}(u) = -\ln(1-u)/\lambda$.

The Gamma Distribution

The Gamma function

Definition 6.14 (Gamma function)

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \text{ for } \alpha > 0.$$

- §6.2 gave the wait for one event. The total of several independent waits needs this function.
- It is a number for each α, not a distribution.
- We state $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ without proof. It needs a double integral, which is beyond this course.

The recursion

Proposition 6.15

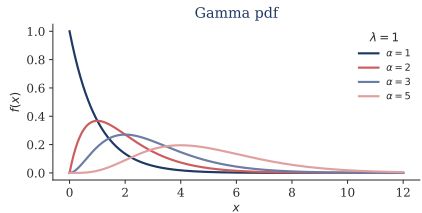
For $\alpha > 0$, $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$. So for a positive integer n, $\Gamma(n) = (n-1)!$.

- The recursion comes from one integration by parts.
- So the Gamma function extends the factorial to every positive number.

The Gamma distribution

Definition 6.16 (Gamma)

$X \sim \text{Gamma}(\alpha, \lambda)$ (伽瑪分配) has $f(x) = \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$, $x > 0$ ($\alpha = 1$ is the exponential).



- It generalizes the exponential. A larger α makes the shape more bell-like.

Gamma MGF, mean, variance

Proposition 6.17

$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$ $(t < \lambda)$; $\mathbb{E}(X) = \frac{\alpha}{\lambda}$, $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$.

- The density integral is a Gamma-function identity. Then differentiate at 0.

Sum of independent exponentials

Proposition 6.18

$X_1, \ldots, X_n$ independent $\text{Exp}(\lambda) \Rightarrow \sum X_i \sim \text{Gamma}(n, \lambda)$.

- MGFs multiply (Prop 4.80), and uniqueness identifies the distribution. The Gamma is to the exponential what the binomial is to the Bernoulli.

Example 6.19

Example 6.19

$X \sim \text{Gamma}(2, 1)$ (sum of two Exp(1)). Find $\mathbb{E}(X)$, $\text{Var}(X)$, $P(X \leq 2)$.

Shapes add

Proposition 6.20

Independent Gamma(α_1, λ) and Gamma(α_2, λ), same λ, sum to Gamma($\alpha_1 + \alpha_2, \lambda$).

- Multiply the MGFs and add the exponents.
- Section 7 uses this. *Chi-square bridge:* $\chi^2_k = \text{Gamma}(k/2, \frac{1}{2})$, so adding independent chi-squares adds their degrees of freedom.

Rate or scale

- Some textbooks do not use the rate λ. They use $\beta = 1/\lambda$, which is called the *scale*.

$$\text{exponential: } f(x) = \frac{1}{\beta} e^{-x/\beta}, \qquad \text{Gamma: } f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}.$$

- These are the same distributions. Only the parameter changes.
- With the scale, the exponential has mean β. The Gamma has mean $\alpha\beta$ and variance $\alpha\beta^2$.
- For example, rate $\lambda = 0.5$ is scale $\beta = 2$.
- We always use the rate. Another book or a software package may use the scale.

Example 6.21

Example 6.21

A repair has three stages. Each stage takes an independent Exp(2) time in hours. Let X be the total repair time.

(a) Name the distribution of X. Find its mean, its variance and its standard deviation.

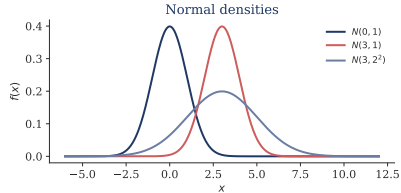
(b) A second repair has two such stages. It is independent of the first repair. Let Y be its total time. Name the distribution of $X + Y$, and find its mean and its variance.

The Normal Distribution

The normal distribution

Definition 6.22 (Normal)

$X \sim N(\mu, \sigma^2)$ (常態分配) has $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$.



- It integrates to 1 by the Gaussian integral (高斯積分), $\int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi}$ (proof beyond scope).

Shape of the normal density

Proposition 6.23

Let $X \sim N(\mu, \sigma^2)$ with density f.

(i) $f(\mu - d) = f(\mu + d)$ for every d. The density is symmetric about μ.

(ii) $f(x) \leq f(\mu)$ for every x. The density is highest at μ.

(iii) $P(X \leq \mu) = 0.5$. So μ is the median of X: half of the probability lies below it.

– The density depends on x only through $(x - \mu)^2$. That gives all three facts.

Normal MGF, mean, variance

Proposition 6.24

$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$; $\mathbb{E}(X) = \mu$, $\mathrm{Var}(X) = \sigma^2$.

- Substitute $x = \mu + \sigma z$, then complete the square.

A linear function of a normal

Proposition 6.25

If $X \sim N(\mu, \sigma^2)$ and $a \neq 0$, then $aX + b \sim N\big(a\mu + b,\ a^2\sigma^2\big)$.

- Section 4 gave the mean and the variance. The new fact is that the result is again normal.
- So a change of units keeps a normal variable normal.

The table we look up

Definition 6.26 (Standard normal)

$Z \sim N(0,1)$ has density $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ and CDF $\Phi(z) = P(Z \le z)$ (appendix table), with $\Phi(-z) = 1 - \Phi(z)$.

- The symmetry rule halves the table. We tabulate $z \ge 0$ only.

Reading the Φ table

Proposition 6.27

Let $Z \sim N(0,1)$ and $a < b$.

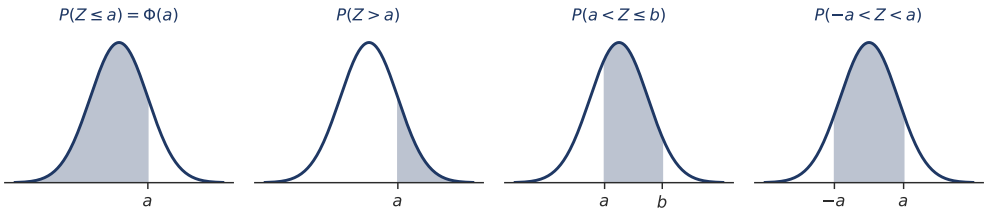
(i) $P(Z > a) = 1 - \Phi(a)$.

(ii) $P(a < Z \leq b) = \Phi(b) - \Phi(a)$.

(iii) For $a > 0$, $P(-a < Z < a) = 2\Phi(a) - 1$.

- The table gives left-tail areas only. Every other area is built from them.

Four areas under the curve



- The table gives the first area directly. Proposition 6.27 turns each of the other three into table entries.

Example 6.28

Example 6.28

Let $Z \sim N(0, 1)$. Use the table to find $P(Z \leq 1.25)$, $P(Z > 0.84)$, $P(-0.50 < Z \leq 1.50)$, and $P(-1.75 < Z < 1.75)$.

One table for the whole family

- The standard normal is the one curve we can actually look up.
- Every normal reduces to it. The recipe has a name: we standardize (標準化) the variable.
- Proposition 6.29 states the recipe and proves that it works.

Reducing any normal to Z

Proposition 6.29

$X \sim N(\mu, \sigma^2) \Rightarrow Z = \dfrac{X - \mu}{\sigma} \sim N(0,1)$, so $P(X \le x) = \Phi\left(\dfrac{x - \mu}{\sigma}\right)$.

- This is Proposition 6.25 with $a = 1/\sigma$ and $b = -\mu/\sigma$.
- There are infinitely many normal distributions. One table serves them all.

Example 6.30

Example 6.30

$X \sim N(100, 15^2)$. Find $P(X < 115)$, $P(85 < X < 130)$, and $P(X > 118)$.

From a probability to a value

- So far each question gave a value and asked for a probability.
- Some questions go the other way. They give a probability p and ask for the value x with $P(X \leq x) = p$.
- This value is the $100p$-th percentile of X.
- To find it, read the table backward. Look for p inside the table, and read z from the margins.

Example 6.31

Example 6.31

(a) $X \sim N(100, 15^2)$. Find the 90th percentile and the 5th percentile. Then find the interval centered at 100 that contains X with probability 0.95.

(b) A machine fills bags of rice. The weight of a bag is $N(\mu, 5^2)$ in grams, and we can set μ. Find the μ for which only 1% of bags weigh less than 500 grams.

Example 6.32

Example 6.32

A fund holds 5 million dollars. Its daily return is $R \sim N(0.1, 2^2)$, in percent. Its daily cost is 2 thousand dollars. Let Y be the profit of one day in thousand dollars.

(a) Write Y as a linear function of R, and find its distribution.

(b) Find the probability that the fund loses more than 100 thousand dollars in a day.

Linear combinations of independent normals

Theorem 6.33

X, Y independent normal, a, b not both $0 \Rightarrow aX + bY \sim N\big(a\mu_X + b\mu_Y,\ a^2\sigma_X^2 + b^2\sigma_Y^2\big)$.

- The MGFs multiply to a normal MGF, and uniqueness identifies it.
- So a portfolio of independent normal returns is itself normal.

Example 6.34

Example 6.34

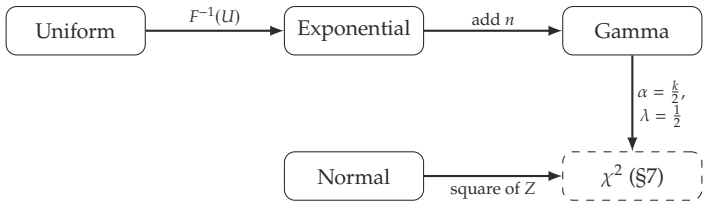
Independent $R_1 \sim N(8, 10^2)$, $R_2 \sim N(12, 20^2)$ (%); portfolio $0.6R_1 + 0.4R_2$. Find its distribution and $P(R_p < 0)$.

Which model?

Ask one question first: **what does X measure?**

X **is**	**Model**
equally likely to be anywhere on an interval	Uniform
the wait until the next event, at a constant rate	Exponential
the total of n independent Exp(λ) waits	Gamma
a total of many small independent effects	Normal

The map


Uniform
$F^{-1}(U)$
Exponential
add n
Gamma
$\alpha = \frac{k}{2}$,
$\lambda = \frac{1}{2}$
Normal
square of Z
χ^2 (§7)