Implied Volatility and the Smile

Financial Engineering · Section 6

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Outline

6.1 Implied volatility
6.2 Why implied volatility matters
6.3 The smile in the data, and what it demands
6.4 The Breeden–Litzenberger formula
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Implied Volatility

Implied volatility

Definition 6.1 (Implied volatility)

Take a quoted call price C^{mkt} with strike K and time to maturity τ, and the observed S and r. The *implied volatility* $\sigma_{\text{imp}}(K, \tau)$ is the σ that solves

$$C^{\text{BS}}(S, K, r, \tau; \sigma) = C^{\text{mkt}}.$$

If the underlying pays a dividend yield δ (Section 4 writes it q), C^{BS} is the Black–Scholes price with yield δ. That is the formula with S replaced by $Se^{-\delta\tau}$. A put's implied volatility is defined in the same way, with the Black–Scholes put price.

- Only σ is not observed. We solve for it.

Why the definition makes sense

Proposition 6.2 (Vega)

Let $\delta = 0$. The Black–Scholes call price is strictly increasing in σ:

$$\mathcal{V} := \frac{\partial C^{\mathrm{BS}}}{\partial \sigma} = S\,\varphi(d_1)\sqrt{\tau} > 0.$$

As σ runs over $(0, \infty)$, C^{BS} takes every value in the interval $\big((S - Ke^{-r\tau})^+,\ S\big)$ exactly once.

- More volatility, higher price.
- The interval holds for any call, in any model.

Proof of Proposition 6.2

Step 1. By $S\varphi(d_1) = Ke^{-r\tau}\varphi(d_2)$ and $d_1 - d_2 = \sigma\sqrt{\tau}$,

$$\frac{\partial C}{\partial \sigma} = S\varphi(d_1)\left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right) = S\varphi(d_1)\sqrt{\tau} > 0.$$

Step 2. $\sigma \to 0$: riskless stock, $C \to (S - Ke^{-r\tau})^+$. $\sigma \to \infty$: $C \to S$.

Step 3. Continuous and strictly increasing: each value once. □

Why Implied Volatility Matters

Three reasons to work in volatility

- **A quoting convention.** It removes strike, maturity and index level.

TXO, strike 23000, 2024-07-01:

maturity	price (points)	implied vol.
2 days	124–143	19.6%
261 days	1290–1310	16.9%

- **A model diagnostic.** Any shape in $\sigma_{\text{imp}}(K, \tau)$ rejects Black–Scholes.
- **A measure of expensiveness.** Implied against realized.

Realized volatility

Definition 6.3 (Realized volatility)

Let $r_1, \ldots, r_n$ be the daily log returns of the underlying over n trading days. The *realized volatility* over these days is

$$\hat{\sigma}_{\mathrm{RV}} = \sqrt{\frac{252}{n} \sum_{i=1}^{n} r_i^2}\,.$$

- The volatility the underlying actually delivered, per year.
- Mean set to zero: tiny at a daily horizon.

Implied against realized

- **A good forecast** of future realized (Christensen and Prabhala, 1998).
- **The gap is a price.** Buy cheap, sell expensive options: significant average returns (Goyal and Saretto, 2009).
- **Index: implied above realized** on average. This is the variance risk premium (Bollerslev, Tauchen and Zhou, 2009).
- Defined in Section 6.7. Proxied by straddle returns in Section 6.6.

The Smile in the Data

Words for the shapes

- Data: TXO (TAIEX index options), the instructor's research panel.
- **Smile**: implied volatility (IV) against strike, one maturity.
- **Smirk**: a smile rising toward low strikes.
- **Wings**: its two ends. **Skew**: its slope.

TXO: a calm day and a crash day

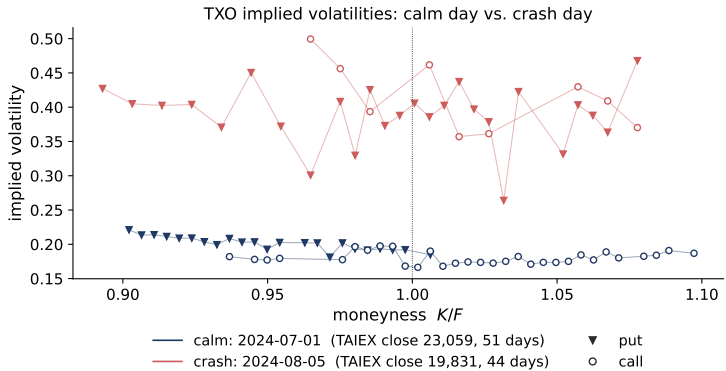


Figure 1.

Reading Figure 1

- Triangles: puts. Circles: calls.
- Moneyness K/F, F the parity forward.
- Calm (51 days): OTM puts show a smirk. 22.1% at 0.90. At the money 17.8% (mean of put and call, as in Figure 2).
- Same strike: put IV often about 2.5 points above the call's. Parity says equal. These quotes break parity. A smile uses OTM only.
- Crash (44 days): both far higher, no clear shape.
- Crash-day slope: not well measured. Forwards are 0.970–0.989 times the close, and the slope changes a lot in that range.

The term structure inverts

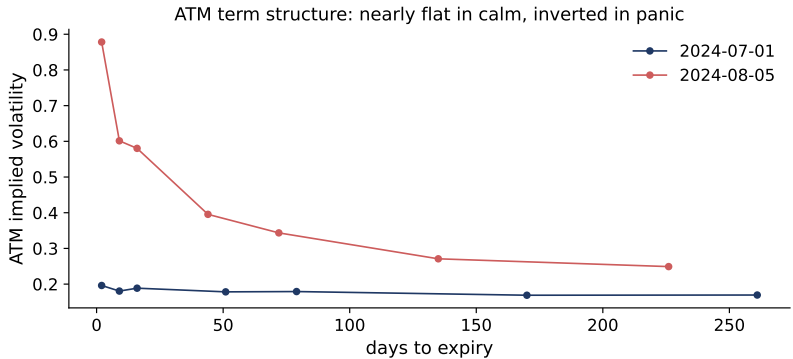


Figure 2.

Reading Figure 2

- Calm: 16.9–19.6%, nearly flat.
- Crash: 87.8% at 2 days, 24.9% at 226 days. Severe but short-lived.

US: VIX and SKEW, 2001–2024

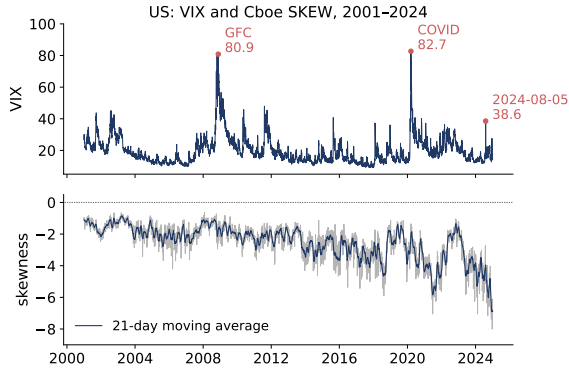


Figure 3.

Reading Figure 3

- Top: the VIX, S&P 500 30-day implied volatility (defined in 6.7). Spikes: 80.9 (2008-11-20), 82.7 (2020-03-16), 38.6 (2024-08-05).
- Bottom: 30-day risk-neutral skewness S, from SKEW $= 100 - 10\,S$.
- Negative on every day: a fat left tail, expensive puts.
- Less negative when the VIX is high: median -1.80 in the highest-VIX fifth of days, -2.69 in the lowest.

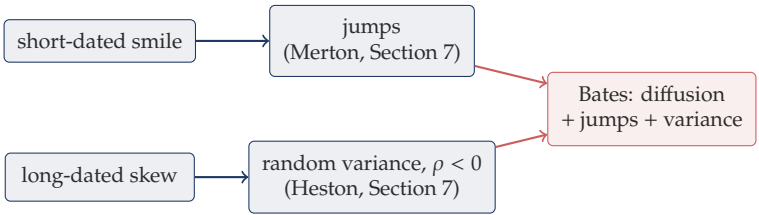
Four stylized facts

1. **The smirk is almost permanent.** The lognormal is rejected.
2. **The level spikes and falls back.** Constant σ is rejected; volatility mean-reverts.
3. **The term structure inverts in stress.** This fits mean-reverting variance.
4. **Short-dated smiles are the most extreme.** A diffusion makes a large two-day move very unlikely. A jump gives it directly.

What the two ends of the term structure demand

- **Short end.** A diffusion moves about $\sigma\sqrt{\Delta t}$. Jumps give large short moves.
- **Long end.** Jumps average out. The skew stays small but not zero (Figure 5).
- Random variance, negatively correlated with returns, keeps it.

What a model of these data needs



– Figure 4. Section 7 builds Bates' model.

The Breeden–Litzenberger Formula

Prices to densities

Theorem 6.4 (Breeden–Litzenberger)

Let $C(K)$ be the time-0 prices of calls of all strikes K with a fixed maturity T. Let q be the risk-neutral density of S_T, continuous at K. Then

$$\frac{\partial C}{\partial K} = -e^{-rT}\,\mathbf{Q}(S_T > K), \qquad \frac{\partial^2 C}{\partial K^2} = e^{-rT}\,q(K).$$

- Slope: the exercise probability. Curvature: the density.
- No model needed.

Butterflies are bets on the density

- Calls: $+1$ at $K-h$, -2 at K, $+1$ at $K+h$:

$$C(K-h) - 2C(K) + C(K+h) \approx e^{-rT} q(K)\, h^2.$$

- Payoff never negative, so $C(K)$ is convex in K.
- $e^{-rT} q(K)\, dK$: the state price near K.

The Implied Volatility Surface

Coordinates for the whole surface

Definition 6.5 (Implied volatility surface)

The *log-moneyness* of strike K at maturity T is $k = \ln(K/F_T)$. The *implied volatility surface* is the function $(k, T) \mapsto \sigma_{\text{imp}}(k, T)$. The *total implied variance* is

$$w(k, T) = \sigma_{\text{imp}}^2(k, T)\, T.$$

- $F_T = Se^{(r-\delta)T}$, the forward.
- w: the variance of $\ln S_T$ that the option prices.

Two numbers price a call

Lemma 6.6 (Black–Scholes in surface coordinates)

Write the normalized call price as $c = C/(e^{-rT}F_T)$. Under Black–Scholes with dividend yield δ,

$$c = \Phi(d_1) - e^k\,\Phi(d_2), \qquad d_{1,2} = -\frac{k}{\sqrt{w}} \pm \frac{\sqrt{w}}{2}.$$

So c depends only on k and w, and it is strictly increasing in w.

- Only moneyness and total variance matter.

Proof of Lemma 6.6

Step 1. Divide $C = Se^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$ by $e^{-rT}F_T = Se^{-\delta T}$. Use $\ln(F_T/K) = -k$ and $\sigma\sqrt{T} = \sqrt{w}$:

$$c = \Phi(d_1) - e^k\Phi(d_2), \qquad d_{1,2} = -\frac{k}{\sqrt{w}} \pm \frac{\sqrt{w}}{2}.$$

Step 2. At fixed k, T: $w = \sigma^2 T$ rises with σ, and the price rises with σ (Proposition 6.2). □

Which surfaces are free of arbitrage

Proposition 6.7 (Static no-arbitrage on the surface)

Suppose call prices are risk-neutral expectations, $C(K,T) = e^{-rT}\mathbb{E}^{Q}[(S_T - K)^+]$, under a measure Q for which $e^{-(r-\delta)t}S_t$ is a martingale. Then:

(a) *Butterfly condition.* At each T, $K \mapsto C(K,T)$ is convex.

(b) *Calendar condition.* At each k, $T \mapsto w(k,T)$ is non-decreasing.

- Only a falling w is an arbitrage, not a falling IV.
- Crash day: ATM w rises. At strike 19200 it falls (Ex. 6.9).

Proof of Proposition 6.7

(a) A butterfly never pays a negative amount. So

$$C(K-h,T)-2C(K,T)+C(K+h,T)\geq 0,$$

which is convexity in K.

(b) Same k, $t<T$, prices in forward units c:

- At t the short option pays its exercise value.
- The long option is still alive: worth at least that.
- So $c(k,T)\geq c(k,t)$, and by Lemma 6.6, $w(k,T)\geq w(k,t)$.

□

Building a surface from quotes

1. **Clean.** Out-of-the-money options only; drop stale quotes.
2. **Find the forward.** Parity: $F_T = K + e^{rT}(C - P)$. F_T/S: [0.997, 1.000] calm, [0.970, 0.989] crash.
3. **Smooth each maturity.** Raw SVI (Gatheral and Jacquier, 2014):

$$w(k) = a + b\left\{\rho\,(k - m) + \sqrt{(k - m)^2 + s^2}\right\}.$$

4. **Join the maturities.** Linear in w at each k.
5. **Check.** Convex prices; w curves do not cross.

Raw prices fail the check

- Convexity broken by over 5 points:
  - calm day: 19 of 207 strike triples;
  - crash day: 36 of 89.
- So the surface must be smoothed.
- SVI: a level, b wings, ρ tilt (not a correlation), m shift, s rounding.

The TXO surface on 2024-07-01

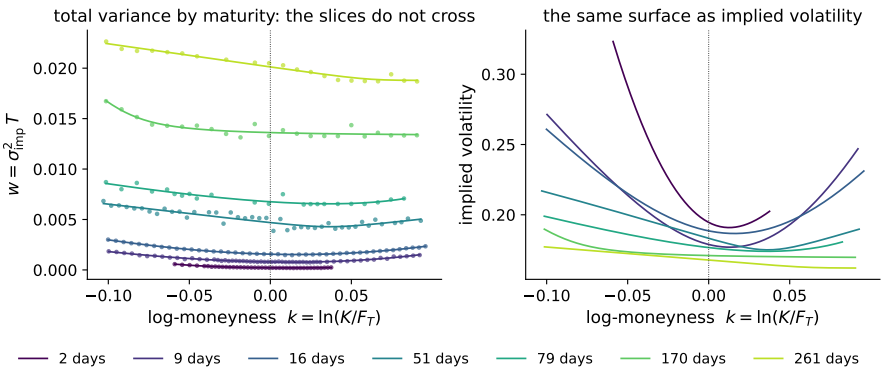


Figure 5.

Reading Figure 5

- SVI fit errors: 0.1–0.6 points. No crossing for $k \in [-0.10, 0.09]$ (2-day and 79-day fits extrapolated there).
- At-the-money slope:

days	2	9–51	79	170–261
slope	about -0.6	-0.23 to -0.33	-0.13	-0.03 to -0.09

- The skew flattens with maturity but does not vanish.

Trading Volatility with Options

Five volatility positions

Definition 6.8 (Five volatility positions)

European options on one underlying, maturity T unless stated, $K_1 < K_2$.

(a) *Straddle* at K: long call and put at K.

(b) *Strangle*: long put at K_1, long call at K_2.

(c) *Butterfly* at K, width h: calls $+1$ at $K - h$, -2 at K, $+1$ at $K + h$.

(d) *Risk reversal*: long call at K_2, short put at K_1.

(e) *Calendar spread* at k: long call at T_2, short call at $T_1 < T_2$, strikes $K_i = F_{T_i} e^k$.

- Each one trades one feature of the surface.

The five positions

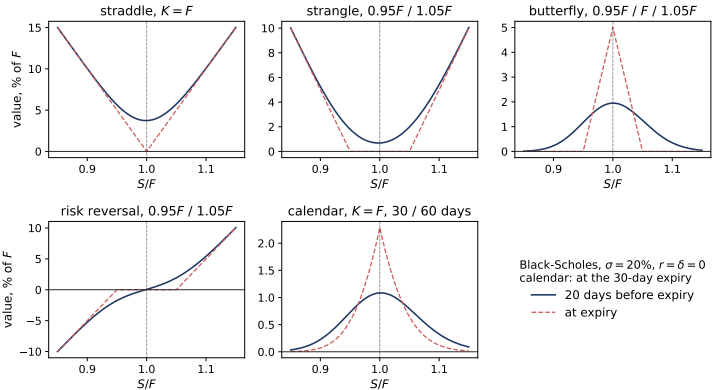


Figure 6.

Which feature each position trades

position	feature
straddle at F_T	level at $k = 0$
strangle	the two tails of q
butterfly	density near K: curvature
risk reversal	skew
calendar spread	term structure of w

Vega on the surface

- Vega: price change per unit change in σ.
- With $K = F_T e^k$, $w = \sigma^2 T$, call and put share it:

$$\mathcal{V}(k) = e^{-rT} F_T \sqrt{T}\, \varphi(d_1), \qquad d_1 = -\frac{k}{\sqrt{w}} + \frac{\sqrt{w}}{2}$$

- Largest at $k = w/2$, near the money.
- Falls to zero in both wings.

Level: the at-the-money straddle

Proposition 6.9 (The at-the-money straddle)

Let $\sigma = \sigma_{\text{imp}}(0, T)$ and $w = \sigma^2 T$. At $K = F_T$ the call and the put have the same price, and the straddle costs

$$\Pi = 2e^{-rT} F_T \left(2\Phi\left(\tfrac{1}{2}\sqrt{w}\right) - 1 \right) \approx e^{-rT} F_T \sqrt{\frac{2}{\pi}}\, \sigma\sqrt{T}.$$

- Price is almost linear in implied volatility.

Proof of Proposition 6.9

Step 1 (call). Lemma 6.6 at $k = 0$: $d_{1,2} = \pm\frac{1}{2}\sqrt{w}$,

$$\frac{C}{e^{-rT}F_T} = \Phi\big(\tfrac{1}{2}\sqrt{w}\big) - \Phi\big(-\tfrac{1}{2}\sqrt{w}\big) = 2\Phi\big(\tfrac{1}{2}\sqrt{w}\big) - 1.$$

Step 2 (put). Parity at $K = F_T$: $C - P = 0$, so $\Pi = 2C$.

Step 3 (Taylor). $\Phi(\pm x) \approx \Phi(0) \pm \varphi(0)x$:

$$2\Phi(x) - 1 \approx 2\varphi(0)\,x = \sqrt{\tfrac{2}{\pi}}\,x, \qquad x = \tfrac{1}{2}\sigma\sqrt{T}.$$

□

The straddle against σ and against S

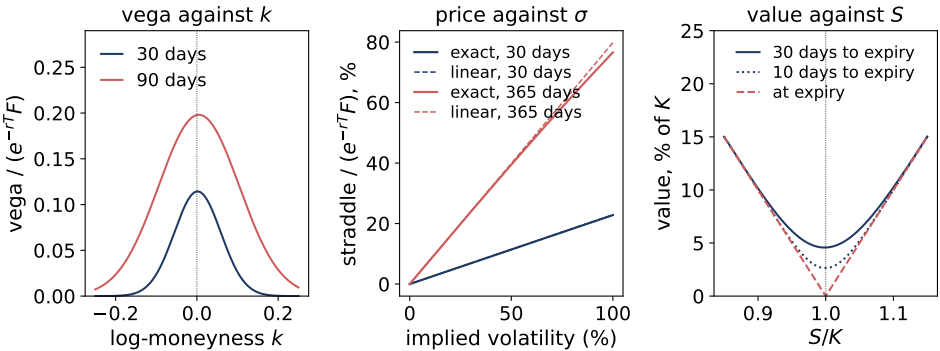


Figure 7.

Reading Figure 7

- Left: vega peaks near the money.
- Middle: exact and linear prices almost coincide.
- Right: far from K, value moves with S.

Tails: strangle and butterfly

- The strangle prices only the tails of q.
- Parity splits the butterfly:

$$\underbrace{C(K-h) - 2C(K) + C(K+h)}_{\text{butterfly} \geq 0} = \underbrace{P(K-h) + C(K+h)}_{\text{strangle}} - \underbrace{\big[C(K) + P(K)\big]}_{\text{straddle}} + e^{-rT}h$$

- Long the wings, short the at-the-money level.
- Gains when the smile curves more.

Skew: the risk reversal

$$\Delta\text{price} \approx \mathcal{V}(k_2)\,\Delta\sigma_{\text{imp}}(k_2) - \mathcal{V}(k_1)\,\Delta\sigma_{\text{imp}}(k_1)$$

- Parallel shift: zero if k_1, k_2 are symmetric about $w/2$.
- Tilt of the smile: gains or loses with the tilt.
- A steeper negative skew hurts the long position.

Term structure: the calendar spread

$$C(K_2, T_2) - C(K_1, T_1) = S\Big[g\big(k, w(k, T_2)\big) - g\big(k, w(k, T_1)\big)\Big] \geq 0$$

- Priced by the variance between T_1 and T_2.
- Gains on a parallel rise.
- Loses when the short end rises much more than the long end (crash day).

The straddle and the variance risk premium

Proposition 6.10 (Expected return of the straddle)

Buy the straddle of Proposition 6.9 at $\sigma_{\text{imp}} = \sigma_{\text{imp}}(0,T)$ and hold it to T. Assume that under $\mathbb{P}$, $\ln(S_T/F_T) \sim N\big(-\tfrac{1}{2}\sigma_R^2 T,\ \sigma_R^2 T\big)$ for a constant $\sigma_R > 0$. Then

$$R = \frac{\mathbb{E}^{\mathbb{P}}|S_T - F_T|}{\Pi} = e^{rT}\,\frac{2\Phi(\tfrac{1}{2}\sigma_R\sqrt{T}) - 1}{2\Phi(\tfrac{1}{2}\sigma_{\text{imp}}\sqrt{T}) - 1} \approx e^{rT}\,\frac{\sigma_R}{\sigma_{\text{imp}}}.$$

- Expected loss exactly when $\sigma_{\text{imp}} > \sigma_R$.

Proof of Proposition 6.10

Step 1. $|S_T - F_T| = (S_T - F_T)^+ + (F_T - S_T)^+$: a straddle payoff.

Step 2. Under $\mathbb{P}$, S_T has the Black–Scholes law with σ_R. So Prop. 6.9 at σ_R, undiscounted:

$$\mathbb{E}^{\mathbb{P}}|S_T - F_T| = 2F_T\big(2\Phi(\tfrac{1}{2}\sigma_R\sqrt{T}) - 1\big).$$

Step 3. Divide by Π. Linearize top and bottom:

$$R \approx e^{rT}\,\frac{\sqrt{2/\pi}\,\frac{1}{2}\sigma_R\sqrt{T}}{\sqrt{2/\pi}\,\frac{1}{2}\sigma_{\text{imp}}\sqrt{T}} = e^{rT}\,\frac{\sigma_R}{\sigma_{\text{imp}}}.$$

□

Reading Proposition 6.10

- Assumes $\mathbb{E}^{\mathbb{P}}[S_T] = F_T$: no premium on the index.
- 20% implied, 16% delivered, 30 days: $R = 0.800$.
- S&P 500 straddles lose on average (Coval and Shumway, 2001).

Simulated straddle returns

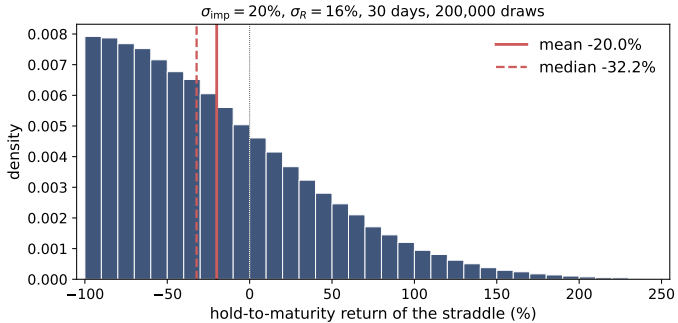


Figure 8. Seed 20261108.

- Mean −20.0%, median −32.2%; 68.1% of trades lose.

TXO straddle returns, 2002–2024

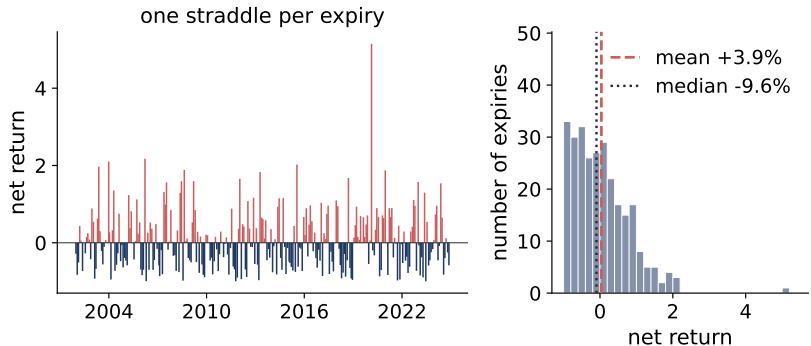


Figure 9.

Reading Figure 9

- 276 monthly trades, about 30 days, held to expiry.
- Largest: +515%, March 2020.
- Median −9.6%; 53.6% of trades lose.
- Mean +3.9%, standard error 0.047: not distinguishable from 0.

TXO: the premium is clear in volatility units

- Mean $\sigma_{\text{imp}} = 19.7\%$, mean $\sigma_R = 17.2\%$.
- $\sigma_R < \sigma_{\text{imp}}$ in 76.8% of months.
- Mean $\sigma_R/\sigma_{\text{imp}} - 1 = -9.6\%$.
- Return vs. this gap: correlation only 0.287.

Why the straddle is a noisy proxy

- Payoff uses the terminal move; σ_R uses the path.
- Fat tails: rare large gains drive the mean.
- One draw per month: only 276 trades.
- Design differs from Coval–Shumway: weekly, zero-beta.

Limits of the straddle as a measure

- Fat tails break the link to σ_R: it prices the size of moves, close to variance but not the same.
- Away from K, value moves with S.
- The delta-hedged straddle is not covered here.

The VIX

The old VIX (1993), now called VXO

- S&P 100 options, two nearest maturities.
- Black–Scholes ATM implied vol from 8 options.
- Trading-day conversion, then 22 trading days:

$$\mathrm{TV} = \mathrm{ATMV}\sqrt{\frac{N_C}{N_T}}, \qquad N_T = N_C - 2\,\mathrm{int}(N_C/7)$$

Carr and Wu (2006).

Four problems with the VXO

- **Model-dependent:** Black–Scholes implied vols.
- **Inflated:** $\sqrt{30/22} \approx 1.17$.
- **Ignores the smile:** only strikes next to the spot.
- **Not replicable:** no static portfolio pays it.

The new VIX (2003)

Definition 6.11 (The VIX formula)

Maturity T, listed strikes $K_1 < \cdots < K_n$. K_0: first listed strike $\le F$. $Q(K_i)$: mid quote of the put below K_0, of the call above, mean of both at K_0. $\Delta K_i = (K_{i+1} - K_{i-1})/2$; at the end strikes, the distance to the one neighbor. Maturities $T_1 < T_2$, normally one on each side of $\tau = 30/365$.

$$\sigma^2(T) = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T}\left(\frac{F}{K_0} - 1\right)^2$$

$$\text{VIX} = 100\sqrt{\frac{w_1 T_1 \sigma^2(T_1) + w_2 T_2 \sigma^2(T_2)}{\tau}}, \quad w_1 = \frac{T_2 - \tau}{T_2 - T_1},\ w_2 = \frac{\tau - T_1}{T_2 - T_1}.$$

- Prices, not implied vols. OTM options only.

Option prices fix expected variance

Proposition 6.12 (Model-free implied variance)

Under $\mathbb{Q}$ let $dS_t/S_t = (r-\delta)\,dt + \sigma_t\,d\widetilde{W}_t$, with σ_t possibly random. Let $C(K)$, $P(K)$ be call and put prices with maturity T, and $Q(K) = P(K)$ for $K < F$, $Q(K) = C(K)$ for $K \geq F$. Then

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \sigma_t^2\,dt\right] = \frac{2e^{rT}}{T}\int_0^\infty \frac{Q(K)}{K^2}\,dK.$$

- Only today's OTM prices. No model for σ_t.

Proof of Proposition 6.12 (1/2): variance is a log contract

Step 1. Itô on $\ln S_t$, integrated:

$$\int_0^T \sigma_t^2 \, dt = 2 \int_0^T \frac{dS_t}{S_t} - 2 \ln \frac{S_T}{S_0}.$$

Take $\mathbb{E}^Q$: $\int dS/S$ has mean $(r - \delta)T$. So

$$\mathbb{E}^Q \left[\int_0^T \sigma_t^2 \, dt \right] = -2 \, \mathbb{E}^Q \left[\ln \frac{S_T}{F} \right].$$

- Expected variance = minus twice a log contract.

Proof of Proposition 6.12 (2/2): the log is a strip of options

Step 2. Exact identity (both sides: 0 at $x = F$, slope $1/x$):

$$\ln \frac{x}{F} = \frac{x-F}{F} - \int_F^x \frac{x-K}{K^2}\, dK$$

$$\Rightarrow \ln \frac{S_T}{F} = \frac{S_T - F}{F} - \int_0^F \frac{(K-S_T)^+}{K^2} dK - \int_F^\infty \frac{(S_T-K)^+}{K^2} dK.$$

Step 3. $\mathbb{E}^Q[S_T] = F$; option payoffs have mean $e^{rT}\times$price. Insert into Step 1. □

Why the weights are $1/K^2$

- One option adds one kink to a payoff.
- Options needed at K = second derivative of the payoff.
- $(\ln K)'' = -1/K^2$: low strikes weigh most.

$$\frac{1}{K^2} \Big/ \frac{1}{F^2} = \begin{cases} 1.56, & K = 0.8F \text{ (put)}, \\ 0.69, & K = 1.2F \text{ (call)}. \end{cases}$$

- The VIX also prices crash protection.

From the integral to the formula

- $K_0 \neq F$: the last term is a small correction.
- Three approximations:
  - the integral becomes a sum over listed strikes;
  - the K_0 term is only the leading term;
  - two maturities are joined linearly in $\sigma^2 T$.

The VIX is a variance swap rate

Definition 6.13 (Variance swap)

A **variance swap** with maturity T and variance strike K_{var} pays at T

$$N \times \left(\text{RV}_{0,T} - K_{\text{var}}\right),$$

with N a notional amount and $\text{RV}_{0,T}$ the realized variance per year over $[0, T]$. In the model of Proposition 6.12, $\text{RV}_{0,T} = \frac{1}{T}\int_0^T \sigma_t^2\, dt$. K_{var} makes the entry cost zero.

- Fair strike = $\mathbb{E}^{Q}[\text{RV}_{0,T}]$ = the option strip.
- $(\text{VIX}/100)^2$: the 30-day variance swap strike, up to the three approximations.
- Downward jumps: VIX^2 slightly below it (Carr and Wu, 2006).

The variance risk premium

Definition 6.14 (Variance risk premium)

$$\mathrm{VRP}_t = \mathbb{E}^{Q}\left[\mathrm{RV}_{t,t+T} \mid \mathcal{F}_t\right] - \mathbb{E}^{\mathbb{P}}\left[\mathrm{RV}_{t,t+T} \mid \mathcal{F}_t\right].$$

For $T = 30$ days the first term is $(\mathrm{VIX}_t/100)^2$, up to the approximations above.

- Long variance pays in crashes: buyers pay a premium.
- VIX about 5 points above realized, 1990–2005 (Carr and Wu, 2006).
- High premium, high next-quarter returns (Bollerslev et al., 2009).

Taiwan's VIX

- TAIFEX, since 18 December 2006.
- Cboe formula on TXO: near and next month.
- F: TX futures price. Updated every 15 seconds.
- Our TXO series: `MF_Vol_30`, BKM (2003), since 2001.

Implied against realized, TXO

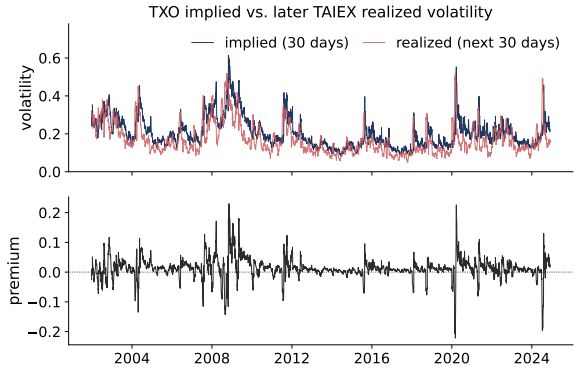


Figure 10.

Reading Figure 10

- 5,019 days, 2001-12-24 to 2024-11-29.
- Implied averages 20.3%, realized 17.0%.
- Premium positive on 79.3% of days.
- Most negative: 2020-03-05, implied 19.8%, realized 51.1%.
- Most positive: 2008-11-07, implied 59.4%, realized 35.1%.

Local Volatility

The smallest change to Black–Scholes

Definition 6.15 (Local volatility model)

In a *local volatility model* the stock follows, under Q,

$$dS_t = rS_t\,dt + \sigma_{\text{loc}}(t, S_t)\,S_t\,d\widetilde{W}_t,$$

where σ_{loc} is a deterministic function of time and the stock price.

- Volatility moves with S and t, but has no randomness of its own.
- One source of risk: a complete market. No dividends here.

Dupire's formula

Proposition 6.16 (Dupire's formula)

In the model of Definition 6.15, let $C(K,T)$ be the call prices of all strikes and maturities. Assume that C is smooth in (K,T) and that $\partial^2 C/\partial K^2 > 0$. Then

$$\sigma^2_{\text{loc}}(T,K) = \frac{\partial C/\partial T + rK\,\partial C/\partial K}{\frac{1}{2}K^2\,\partial^2 C/\partial K^2}.$$

- Today's prices give the local variance. Not proved here (Dupire, 1994).

The formula is two trades

- Denominator: a **butterfly**, the density at K.
- Numerator: a **calendar spread** at fixed moneyness k.
- No static arbitrage (Proposition 6.7): both are ≥ 0.
- So the local variance is not negative.
- Arbitrage-free surface: it fixes σ_{loc}, which fits every price.

An exact fit, but the wrong dynamics

- A fixed $\sigma_{\text{loc}}(t, S)$ predicts how the smile moves.
- It moves the wrong way (Hagan et al., 2002).
- So its hedges are wrong. Section 7 adds random volatility.

Using the Surface

The slope of the smile prices a digital

Proposition 6.17 (The digital call off the smile)

Let $\delta = 0$ and fix a maturity T. Let the market call prices be $C(K) = C^{\mathrm{BS}}\big(K, \sigma_{\mathrm{imp}}(K)\big)$, with σ_{imp} differentiable in K. Then the digital call with strike K costs

$$D(K) = e^{-rT}\,\Phi(d_2) - \mathcal{V}\,\frac{\partial \sigma_{\mathrm{imp}}}{\partial K},$$

where d_2 and the vega $\mathcal{V} = S\varphi(d_1)\sqrt{T}$ (Proposition 6.2) are evaluated at $\sigma_{\mathrm{imp}}(K)$.

- Negative skew: the digital costs more than the flat-smile price.

Proof of Proposition 6.17 (1/2)

Step 1 (a strike derivative). The digital pays $\mathbf{1}\{S_T > K\}$, so by Theorem 6.4

$$D(K) = e^{-rT}\mathbb{Q}(S_T > K) = -\frac{\partial C}{\partial K}.$$

Step 2 (chain rule).

$$\frac{\partial C}{\partial K} = \frac{\partial C^{\mathrm{BS}}}{\partial K} + \frac{\partial C^{\mathrm{BS}}}{\partial \sigma}\,\frac{\partial \sigma_{\mathrm{imp}}}{\partial K} = \frac{\partial C^{\mathrm{BS}}}{\partial K} + \mathcal{V}\,\frac{\partial \sigma_{\mathrm{imp}}}{\partial K}.$$

Proof of Proposition 6.17 (2/2)

Step 3 (the Black–Scholes strike derivative). Since $\partial d_1/\partial K = \partial d_2/\partial K$ and $S\varphi(d_1) = Ke^{-rT}\varphi(d_2)$,

$$\frac{\partial C^{\mathrm{BS}}}{\partial K} = S\varphi(d_1)\frac{\partial d_1}{\partial K} - e^{-rT}\Phi(d_2) - Ke^{-rT}\varphi(d_2)\frac{\partial d_2}{\partial K} = -e^{-rT}\Phi(d_2).$$

Put Steps 2 and 3 into Step 1. □

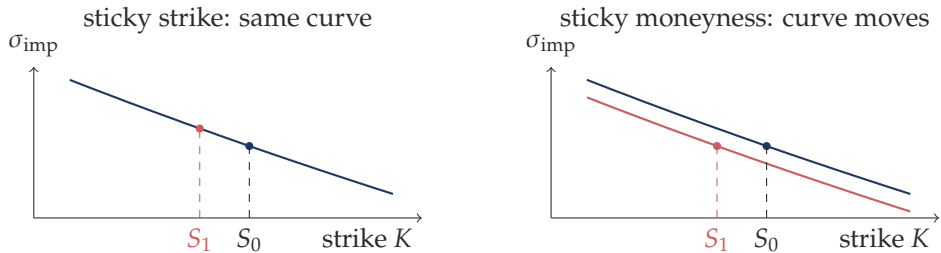
How the surface moves

Definition 6.18 (Sticky strike and sticky moneyness)

When the spot moves, the *sticky-strike* rule keeps the implied volatility of each strike K unchanged. The *sticky-moneyness* rule keeps the implied volatility of each moneyness K/S unchanged.

- Sticky strike: the smile stays. Sticky moneyness: it moves with S.
- Each fits better in some periods (Derman, 1999).

The two rules after a fall in the spot



- Figure 11. Left: new volatility higher up the same curve. Right: unchanged.
- Local volatility moves the smile against the spot.

Risk on the surface: vega buckets

- **Buckets**: vega by maturity and by moneyness.
- Flat to a parallel shift, yet a loss in a twist (Figure 2).

The surface as a signal

- **Crash risk.** S&P 500 skewness negative on every day, 2001–2024. Bates (1991).
- **Stock returns.** Steep smirk, lower returns (Xing et al., 2010).
- **Option returns.** Goyal and Saretto (2009).
- **Demand.** Buying pressure shapes the smile (Bollen and Whaley, 2004).
- **Whole surfaces.** Machine learning (Kelly et al., 2023).