Computational Methods

Financial Engineering · Section 5

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Fall 2026

Outline

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5.2 Symbols, then N periods
5.3 Calibrating the tree to Black–Scholes
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Two Periods, Then Three

The model

- One stock has price S at a node. Over one period it moves to uS (up) or to dS (down), with $0 < d < u$.
- Cut $[0, T]$ into N periods of length $\Delta t = T/N$ (one symbol, not the hedge Δ times t). The account pays r per year, so it grows by $1 + r\Delta t$ over one period.
- Δt is the period length in years: $r\Delta t$ is one period's interest. With T fixed, more periods mean a smaller Δt.
- Shares can be bought or shorted in any fraction, and the account lends and borrows at this rate.
- The method: the seller holds Δ shares against one call, so that the position is riskless. Do this node by node, from the last date back to date 0.

Example 5.1

Example 5.1

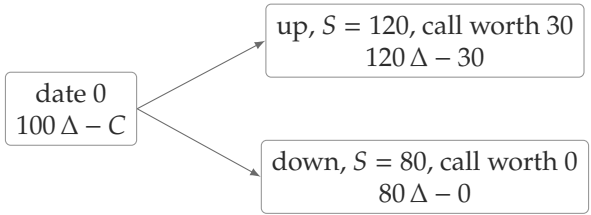
Take $S_0 = 100$, $u = 1.2$, $d = 0.8$, $r = 10\%$ per year, $T = 2$ years, $N = 2$ (so $\Delta t = 1$), and the call with $K = 100$. The seller writes one call and holds Δ shares against it.

(a) Draw the stock tree and the call's payoffs at date 2.

(b) At each date-1 node, find the Δ that makes the seller's position riskless over the last period. What is the call worth at that node?

(c) Do the same at date 0. What is the call worth today?

The seller's view



- Sell one call at price C, buy Δ shares: $100\,\Delta - C$ today.
- Riskless when both outcomes agree: $120\,\Delta - 30 = 80\,\Delta$, so $\Delta = \frac{3}{4}$ and the position pays 60 for sure.
- A sure 60 must cost $60/1.1 = 54.5455$ today, or there is an arbitrage against the account. So $75 - C = 54.5455$ and $C = 20.4545$.

The hedge needs no outside money

- Date 0: receive 20.4545, buy $\frac{3}{4}$ of a share for 75, borrow the other 54.5455.
- Up move: shares 90, loan 60, so the seller has $30 = C_u$. Buy $\frac{1}{6}$ of a share more for 20, with a further loan of 20.
- Date 2: the loan is 88. Up-up: $132 - 88 - 44 = 0$. Up-down: $88 - 88 = 0$.
- Down move: shares 60, loan 60. Nothing is left or owed.
- No outside money enters after date 0. The hedge is self-financing.

Example 5.2

Example 5.2

Same tree as Example 5.1: $S_0 = 100$, $u = 1.2$, $d = 0.8$, $r = 10\%$ per year, $\Delta t = 1$, $K = 100$, now with $T = 3$ years and $N = 3$. Price the European call by the seller's hedge, node by node.

Symbols, Then N Periods

The hedge at a node

- Stock S, successor values V^u, V^d. The seller is short the claim and holds Δ shares. Riskless when

$$\Delta\, uS - V^u = \Delta\, dS - V^d, \qquad \Delta = \frac{V^u - V^d}{(u-d)S}.$$

- Δ is the spread of the successor values over the spread of the successor prices.
- The sure amount is $A = \Delta\, dS - V^d = \dfrac{dV^u - uV^d}{u-d}$, so $\Delta S - V = \dfrac{A}{1 + r\Delta t}$.

The roll-back rule

- Solve for V. Two fractions appear, and they add up to 1:

$$V = \frac{\dfrac{1 + r\Delta t - d}{u - d} V^u + \dfrac{u - 1 - r\Delta t}{u - d} V^d}{1 + r\Delta t}.$$

- Call the first one q. Then

$$q = \frac{1 + r\Delta t - d}{u - d}, \qquad V = \frac{q\, V^u + (1 - q)\, V^d}{1 + r\Delta t}.$$

- Value equals the q-average of next period's values, discounted one period. The same q works at every node and for every claim.
- Check: $q = \frac{3}{4}$, $30 = (\frac{3}{4} \cdot 44 + \frac{1}{4} \cdot 0)/1.1$, $20.4545 = (\frac{3}{4} \cdot 30 + \frac{1}{4} \cdot 0)/1.1$.

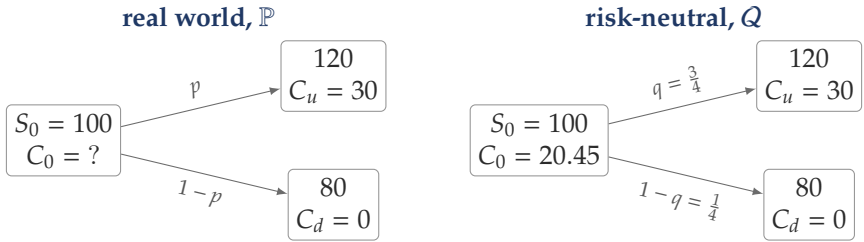
The binomial specimen

Proposition 5.3 (The binomial specimen)

In the binomial model with up factor u, down factor $d < u$, rate r and period length Δt: the market is arbitrage-free if and only if $d < 1 + r\Delta t < u$; in that case the equivalent martingale measure Q is unique, given at every node by the one-period probability $q = \frac{(1+r\Delta t)-d}{u-d}$, and the market is complete: every claim is replicated by a self-financing strategy computed by backward induction.

- If $1 + r\Delta t \leq d$ the stock dominates. If $1 + r\Delta t \geq u$ the account dominates. Either way there is an arbitrage.
- Exactly one reweighting makes the discounted stock a martingale. q is a pricing weight, not a forecast. On the tree of Example 5.1, $q = \frac{3}{4}$, and this is not Δ_0.

Why q is called risk-neutral



- Under $q = \frac{3}{4}$, $1 - q = \frac{1}{4}$ every asset earns the riskless rate on average: $\frac{3}{4}(120) + \frac{1}{4}(80) = 110 = 100 \times 1.1$ and $\frac{3}{4}(30) + \frac{1}{4}(0) = 22.5 = \frac{225}{11} \times 1.1$.
- Averaging with the real p would need the option's own required return as the discount rate, and nobody knows it. With the risk-neutral weights we discount at the known rate r.

Two periods in symbols

- f is the value at date 0, f_u, f_d the values at date 1, and f_{uu}, f_{ud}, f_{dd} the payoffs at date 2.
- $f_u = \dfrac{qf_{uu} + (1-q)f_{ud}}{1 + r\Delta t}$, $\quad f_d = \dfrac{qf_{ud} + (1-q)f_{dd}}{1 + r\Delta t}$, and

$$f = \frac{qf_u + (1-q)f_d}{1 + r\Delta t} = \frac{q^2 f_{uu} + 2q(1-q)f_{ud} + (1-q)^2 f_{dd}}{(1 + r\Delta t)^2}.$$

- Check on Example 5.1: $\left(\frac{9}{16}(44)\right)/1.21 = 24.75/1.21 = 20.4545$. Same q at every node, because u, d, Δt, r are the same at every node.

Pricing on N periods

Proposition 5.4 (Pricing on N periods)

Consider the N-period binomial model with $d < 1 + r\Delta t < u$ and $q = (1 + r\Delta t - d)/(u - d)$. Write S_k for the stock price at date k, and let a claim pay $h(S_N)$ at date N.

(a) (Backward induction.) Its price V_k at every node of date k is given by $V_N = h(S_N)$ and $V_k = \big(q\,V^u_{k+1} + (1-q)\,V^d_{k+1}\big)/(1 + r\Delta t)$, $k = N-1, \dots, 0$, where V^u_{k+1}, V^d_{k+1} are the values at the two successor nodes.

(b) (Closed form.) The date-0 price is $V_0 = (1 + r\Delta t)^{-N} \sum_{j=0}^{N} \binom{N}{j} q^j (1-q)^{N-j}\, h\big(S_0 u^j d^{N-j}\big)$.

(c) (Hedge.) At every node the replicating portfolio holds $\Delta = (V^u_{k+1} - V^d_{k+1})/\big((u-d)S_k\big)$ shares, and $b = V_k - \Delta S_k$ in the riskless account.

– (a) is the roll-back rule applied $1 + 2 + \cdots + N$ times; (b) is the same price in one formula, since the number of up moves is Binomial(N, q).

The call in two tails

Proposition 5.4(d) (The call in two tails)

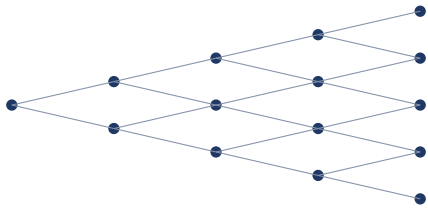
For the call, $h(S) = (S - K)^+$, let m be the smallest non-negative integer with $S_0 u^m d^{N-m} \geq K$. Then

$$C_0 = S_0 \widetilde{\boldsymbol{Q}}(J \geq m) - K(1 + r\Delta t)^{-N} \boldsymbol{Q}(J \geq m),$$

where J is the number of up moves, $J \sim \text{Binomial}(N, q)$ under $\boldsymbol{Q}$ and $J \sim \text{Binomial}(N, \tilde{q})$ under $\widetilde{\boldsymbol{Q}}$, with $\tilde{q} = qu/(1 + r\Delta t)$. (If no such $m \leq N$ exists, the call is worth 0.)

- In the money exactly when at least m moves are up: discounted strike times its $\boldsymbol{Q}$-probability, stock price times its $\widetilde{\boldsymbol{Q}}$-probability.
- Compare $C = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$: see Theorem 5.6.

A recombining lattice



$k + 1$ nodes at date k, not 2^k paths

- Only the number of up moves matters, not their order. With $N = 500$ periods: 125,751 nodes, not 2^{500} paths.

Calibrating the Tree to Black–Scholes

What one period must match

- Divide $[0, T]$ into N periods of length $\Delta t = T/N$. The unknowns are u, d, q.
- One Black–Scholes period, under Q:

$$S_{t+\Delta t} = S_t \exp\Big(\big(r - \tfrac{\sigma^2}{2}\big)\Delta t + \sigma\sqrt{\Delta t}\, Z \Big), \qquad Z \sim N(0,1).$$

- Black–Scholes compounds continuously, so the growth is $e^{r\Delta t}$ in place of $1 + r\Delta t$: $e^{r\Delta t} = 1 + r\Delta t + O((\Delta t)^2)$.
- Take q as in Proposition 5.3 with this factor: $q = (e^{r\Delta t} - d)/(u - d)$. This fixes the mean. Then $ud = 1$ keeps the lattice centered, and the variance fixes u to leading order.

Proposition 5.5, one period of each model

Proposition 5.5 (Matching one period: the CRR choice)

Let $d < e^{r\Delta t} < u$ and $q = (e^{r\Delta t} - d)/(u - d)$. All moments are those of $S_{t+\Delta t}$ under Q, conditional on S_t. Every $O(\cdot)$ is for $\Delta t \to 0$ with r and σ fixed.

(a) (Black–Scholes period.) The mean is $S_t e^{r\Delta t}$. The variance is $S_t^2 e^{2r\Delta t}(e^{\sigma^2\Delta t} - 1) = S_t^2\big(\sigma^2\Delta t + O((\Delta t)^2)\big)$.

(b) (Tree period.) The mean is $S_t e^{r\Delta t}$, exactly. The variance is $S_t^2\, q(1-q)(u-d)^2 = S_t^2\big[e^{r\Delta t}(u+d) - ud - e^{2r\Delta t}\big]$.

– The two means agree exactly. Only the variance is left to match.

Proposition 5.5, the choice

Proposition 5.5 (continued)

(c) (The choice.) If $ud = 1$ and the two variances, divided by S_t^2, agree up to $O((\Delta t)^2)$, then $u + 1/u = 2 + \sigma^2 \Delta t + O((\Delta t)^2)$, so $\ln u = \sigma\sqrt{\Delta t} + O((\Delta t)^{3/2})$. The Cox–Ross–Rubinstein (CRR) choice drops the remainder:

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = \frac{1}{u}, \qquad q = \frac{e^{r\Delta t} - d}{u - d},$$

with discounting at $e^{-r\Delta t}$ per period. Its variance is $S_t^2\big(\sigma^2 \Delta t + O((\Delta t)^2)\big)$.

(d) (The weight.) As $\Delta t \to 0$, the CRR q satisfies $q = \frac{1}{2} + c\sqrt{\Delta t} + O(\Delta t)$, with $c = (r - \sigma^2/2)/(2\sigma)$.

Why CRR is the right tree

- The mean is right in every period, exactly. The variance is right up to $O((\Delta t)^2)$. Over $N = T/\Delta t$ periods the errors add up to $O(\Delta t)$.
- Interest per period is of order Δt. The stock's log move is of order $\sqrt{\Delta t}$, so its square is of order Δt. Over N periods the sums are about rT and $\sigma^2 T$. The total variance stays finite and positive.
- The up factor is a one-standard-deviation move, the $\sqrt{\Delta t}$ scaling of §3.1.
- Not the only choice: Jarrow and Rudd (1983) fix $q = \frac{1}{2}$ and put the drift into u and d. Same limit.

The tree becomes the formula

Theorem 5.6 (CRR → Black–Scholes)

Let C_N be the N-period CRR price of the European call (strike K, maturity T). Then

$$C_N \longrightarrow C_{\mathrm{BS}} = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \qquad (N \to \infty).$$

- As the periods shrink, the tree price becomes the Black–Scholes price.
- The observed error is $O(1/N)$, so doubling the periods halves it.

How fast: the ATM benchmark

$S_0 = K = 100$, $r = 0$, $\sigma = 0.2$, $T = 1$; Black–Scholes price 7.9656.

N	C_N	error	$N\times$ error
10	7.7692	−0.1963	−1.96
25	8.0451	+0.0795	+1.99
50	7.9258	−0.0397	−1.99
100	7.9457	−0.0199	−1.99
250	7.9576	−0.0080	−1.99
500	7.9616	−0.0040	−1.99

- $N\times$ error is constant: the error is $O(1/N)$.
- The sign flips with N: the odd–even oscillation. Average two adjacent N rather than just increasing N.

American Options on the Tree

Exercise at any date

- A European option is exercised only at date N. An American option can be exercised at any date $k = 0, 1, \ldots, N$, once.
- Exercising at date k pays the exercise value $h(S_k)$: $(K - S_k)^+$ for a put, $(S_k - K)^+$ for a call.
- At every node the holder compares: exercise now, or hold one more period. The value of holding is the roll-back value, the continuation value.
- The option is worth the larger of the two. The tree handles this with one extra operation per node.

The American tree price

Definition 5.7 (American option and its tree price)

An American option with exercise value $h(\cdot)$ gives its holder the right to receive $h(S_k)$ at any one date $k \in \{0, 1, \ldots, N\}$ of her choice. Its price on the N-period binomial tree is defined by backward induction:

$$V_N = h(S_N), \qquad V_k = \max\left\{h(S_k),\ \frac{q\,V^u_{k+1} + (1-q)\,V^d_{k+1}}{1 + r\Delta t}\right\}, \qquad k = N-1, \ldots, 0.$$

A node where $h(S_k)$ is strictly larger than the continuation value is an early-exercise node: the holder exercises there.

- One maximum added to the roll-back rule; interest creates early exercise.

Example 5.8

Example 5.8

Same tree as Example 5.1: $S_0 = 100$, $u = 1.2$, $d = 0.8$, $r = 10\%$ per year, $\Delta t = 1$, $T = 2$ years, $N = 2$. Price the European put and the American put with $K = 100$. Say at which node, if any, the American put is exercised early.

The call is never exercised early

Proposition 5.9 (No early exercise of the call)

In the N-period binomial model with $r \geq 0$ and a stock that pays no dividends, the tree price of the American call with strike K equals the tree price of the European call at every node. No node is an early-exercise node.

- Paying K later is cheaper when $r \geq 0$, and holding keeps the chance that the stock rises further. So the early-exercise right of a call is worth zero.
- For the put the comparison runs the other way, and that is where the extra value of the American put in Example 5.8 comes from.

Monte Carlo Methods

Tree versus simulation

- **Tree:** backward induction from maturity. Converges to Black–Scholes by the central limit theorem (Theorem 5.6). Handles early exercise with one maximum.
- **Monte Carlo:** forward simulation from today. Converges by the law of large numbers. Handles path-dependent payoffs and many underlyings, where a tree becomes too large to build.
- **Random numbers:** a seeded recursion gives Uniform$(0, 1)$ draws; Box–Muller turns two of them into two independent $N(0, 1)$ draws, $Z_1 = \sqrt{-2 \ln U_1} \cos(2\pi U_2)$, $Z_2 = \sqrt{-2 \ln U_1} \sin(2\pi U_2)$ (Exercise 5.14). Same seed, same simulation.

Monte Carlo

- Draw N independent copies of S_T under Q and average (here N counts paths, not periods):

$$\widehat{C}_N = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} h\big(S_T^{(i)}\big), \qquad \widehat{C}_N \approx C \pm z_{\alpha/2}\frac{s}{\sqrt{N}}.$$

- The rate $N^{-1/2}$ contains no dimension. Pricing on five underlyings costs the same order of work as on one. So Monte Carlo is the standard method for multi-asset and path-dependent claims.
- The cost is the exponent $1/2$: each extra decimal digit of accuracy needs a hundred times more paths.

Exact sampling, and when you cannot have it

- For GBM no discretization is needed. Sample $Z \sim N(0,1)$ and set $S_T = S_0 \exp\big((r - \sigma^2/2)T + \sigma\sqrt{T}\,Z\big)$. This is *exact*.
- Without a closed-form terminal law (Heston, Section 7) simulate the SDE on a grid by the Euler–Maruyama step $X_{t+\Delta t} = X_t + \mu(t, X_t)\Delta t + \sigma(t, X_t)\sqrt{\Delta t}\,Z$.
- That introduces a discretization bias, weak error $O(\Delta t)$, which then adds to the statistical error. Two error sources, not one.

A European call in four steps

1. Draw N independent standard normals $Z^{(1)}, \dots, Z^{(N)}$.
2. Terminal prices: $S_T^{(i)} = S_0 \exp\big((r - \sigma^2/2)T + \sigma\sqrt{T}\, Z^{(i)}\big)$.
3. Discounted payoffs: $C^{(i)} = e^{-rT}\big(S_T^{(i)} - K\big)^+$.
4. Average: $\widehat{C}_N = \frac{1}{N}\sum_{i=1}^{N} C^{(i)}$, reported with its standard error $s/\sqrt{N}$.

- A Monte Carlo estimate without its standard error is not a result.
- Two correlated underlyings: draw independent Z_1, Z_2 and set $\varepsilon_1 = Z_1$, $\varepsilon_2 = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2$. Then $\mathrm{Var}(\varepsilon_2) = 1$ and $\mathrm{Corr}(\varepsilon_1, \varepsilon_2) = \rho$: the Cholesky factorization of the 2×2 correlation matrix.

Example 5.10

Example 5.10

The accuracy budget. For the ATM benchmark, compute $\mathrm{Var}^{Q}\left[(S_T - K)^+\right]$ analytically and determine how many paths a 1-cent standard error requires. Use $\Phi(0.3) = 0.617911$, $\Phi(0.1) = 0.539828$, $e^{0.04} = 1.040811$ (six decimals: the factor 10^4 uses four, and the subtraction cancels two leading digits) and $\mathbb{E}^{Q}[S_T^2 \mathbf{1}\{S_T > K\}] = S_0^2 e^{(2r+\sigma^2)T}\Phi(d_1 + \sigma\sqrt{T})$.

Variance reduction

- **Antithetic variates.** Use Z and $-Z$ in pairs:

$$\mathrm{Var}\left(\frac{f(Z)+f(-Z)}{2}\right) = \frac{\mathrm{Var}(f)}{2}(1+\rho), \qquad \rho = \mathrm{Corr}(f(Z), f(-Z)).$$

  Wins when $\rho < 0$: monotone payoffs. Nearly linear payoffs gain most.
- **Control variates.** Simulate g with known $\mathbb{E}[g]$ alongside f and correct by its observed error:

$$\widehat{C}^{\mathrm{cv}} = \bar{f} - \beta(\bar{g} - \mathbb{E}[g]), \qquad \beta^* = \frac{\mathrm{Cov}(f,g)}{\mathrm{Var}(g)},$$

  which multiplies the variance by $1 - \rho_{f,g}^2$. Variance reduction is a regression of the payoff on a quantity whose mean is known.

Least-Squares Monte Carlo

Why plain Monte Carlo cannot price American options

- The tree needs a continuation value at every node: a conditional expectation of next period's value given S_k.
- The tree has it because every node branches into the same two successors.
- A simulated path does not branch. At date t on path i there is one future, the rest of path i. Its later payoff is one draw, not an expectation.
- Longstaff and Schwartz (2001): estimate the continuation value by a regression across paths of the later payoff Y on functions of the current price X.
- Why it works: $\mathbb{E}[Y \mid X]$ minimizes $\mathbb{E}[(Y - g(X))^2]$ over all g; least squares does the same with sample averages and $g = a + bx + cx^2$.

The LSM algorithm

1. Simulate N paths under Q on the exercise dates $t_1 < \cdots < t_M = T$.
2. At T set the cash flow of path i to $(K - S_T^{(i)})^+$.
3. Go backward, $t_m = t_{M-1}, \ldots, t_1$:
   (a) select the paths in the money at t_m;
   (b) $X^{(i)} = S_{t_m}^{(i)}$, $Y^{(i)}$ = the path's later cash flow discounted to t_m;
   (c) regress Y on $1, X, X^2$; the fitted value is the estimated continuation value;
   (d) if $K - X^{(i)}$ exceeds it, exercise: cash flow $K - X^{(i)}$ at t_m, later cash flows set to zero.
4. Price = average over paths of the (single) cash flow discounted to date 0.

Example 5.11

Example 5.11

American put, $K = 1.10$, $S_0 = 1$, exercise dates $t = 1, 2, 3$, $r = 6\%$ per year, $\Delta t = 1$, continuously compounded ($e^{-0.06} = 0.94176$). The eight paths of Longstaff and Schwartz (2001):

path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

Example 5.11, the decision at $t = 2$

- Cash flows at $t = 3$: $0, 0, 0.07, 0.18, 0, 0.20, 0.09, 0$.
- In the money at $t = 2$: paths 1, 3, 4, 6, 7. $Y = 0.94176\times$ the $t = 3$ cash flow.

path	1	3	4	6	7
$X = S_2$	1.08	1.07	0.97	0.77	0.84
Y	0	0.0659	0.1695	0.1884	0.0848
exercise $1.10 - X$	0.02	0.03	0.13	0.33	0.26
continuation	0.0367	0.0459	0.1175	0.1520	0.1564
decision	hold	hold	exercise	exercise	exercise

- Least squares on $1, X, X^2$ (exact sums; the system is badly conditioned): $\widehat{\text{Cont}}(X) = -1.0691 + 2.9817X - 1.8127X^2$.
- Paths 4, 6, 7 exercise at $t = 2$; their $t = 3$ cash flows become 0.

Example 5.11, the decision at $t = 1$

- In the money at $t = 1$: paths 1, 4, 6, 7, 8. Y = later cash flow discounted to $t = 1$ (paths 1 and 8 have none).

path	1	4	6	7	8
$X = S_1$	1.09	0.93	0.76	0.92	0.88
Y	0	0.1224	0.3108	0.2449	0
exercise $1.10 - X$	0.01	0.17	0.34	0.18	0.22
continuation	0.0135	0.1088	0.2861	0.1170	0.1528
decision	hold	exercise	exercise	exercise	exercise

- Least squares on $1, X, X^2$: $\widehat{\text{Cont}}(X) = 2.0377 - 3.3359X + 1.3567X^2$.
- Paths 4, 6, 7, 8 exercise at $t = 1$.

Example 5.11, the price

- Final cash flows: paths 4, 6, 7, 8 at $t = 1$ (0.17, 0.34, 0.18, 0.22); path 3 at $t = 3$ (0.07); paths 1, 2, 5 nothing.
- Discount to date 0 and average:

$$\widehat{V}_0 = \frac{(0.17 + 0.34 + 0.18 + 0.22)(0.94176) + (0.07)(0.83527)}{8}$$

$$= \frac{0.8570 + 0.0585}{8} = 0.1144.$$

- The European put on the same eight paths: $\dfrac{0.54 \times 0.83527}{8} = 0.0564.$
- The early-exercise right doubles the value on these paths. In practice N is in the tens of thousands, and the regression at each date has thousands of points.