The Black–Scholes World

Financial Engineering · Section 4

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The Black–Scholes Assumptions and Ideas

The five assumptions

- **A1. Constant volatility.** σ is a known constant. So S_T is lognormal, and there is one number to calibrate.
- **A2. One constant rate.** Borrowing and lending at the same r. One discount factor, $e^{-r\tau}$, with $\tau = T - t$.
- **A3. No dividends.** The stock pays nothing before T. §4.8 removes this.
- **A4. Frictionless continuous trading.** No costs, no taxes, any fraction of a share at any instant. This makes exact replication possible.
- **A5. Continuous paths.** The stock is a geometric Brownian motion. It never jumps, so a hedge can be adjusted through every move.

Ideas 1 and 2: a redundant asset, priced by no arbitrage

- **The option is a redundant asset.** Its payoff $h(S_T)$ can be reproduced by a position in the stock and the money market. The position is adjusted continuously, with no money added or withdrawn. §4.3 constructs it.
- **The price is fixed by no arbitrage alone.** If the option traded above the cost of the reproducing position, sell the option and buy the position. So the option costs what the position costs. No risk preference enters.

Idea 3: the price is a function $C(t, S)$

- By the GBM solution of §3.4,

$$S_T = S_t \, e^{(\mu - \sigma^2/2)(T-t) + \sigma(W_T - W_t)},$$

  and $W_T - W_t$ is independent of everything known at t (§3.1).
- So the date t and the current price S_t are all the option's value can depend on.
- Everything in this section is a statement about the function $C(t, S)$.

Ideas 4 and 5: μ drops out, and one price has three descriptions

- **μ plays no role and σ does.** Two investors may disagree about the expected return but agree on the volatility. Then they agree on every option price. §4.2 shows where μ disappears.
- **Three descriptions of one price.** C solves a PDE (§4.3). C is a discounted expectation under a special measure (§4.4). C is the value of the reproducing position (§4.3). The section walks through them in that order.

The market

Definition 4.1 (Black–Scholes market)

Under $\mathbb{P}$:

- **a stock** $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$, $\sigma > 0$, solved in §3.4 as $S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t}$;
- **a money market account** $B_t = e^{rt}$, so $dB_t = rB_t\,dt$.

A portfolio $V_t = a_t S_t + b_t B_t$, with a, b adapted, is *self-financing* if $dV_t = a_t\,dS_t + b_t\,dB_t$.

- Every change in value comes from price moves, none from money added or withdrawn. This is Section 2's self-financing definition, with the adjustments now continuous.

The Stock under the Risk-Neutral Measure

The discounted stock under $\mathbb{P}$

- Discount: $\widetilde{S}_t = S_t/B_t$. The factor $1/B_t$ is deterministic, so the product rule of §3.5 has an empty bracket term:

$$d\widetilde{S}_t = \widetilde{S}_t\big((\mu - r)\,dt + \sigma\,dW_t\big) = \sigma\widetilde{S}_t\Big(\underbrace{\tfrac{\mu - r}{\sigma}}_{\lambda}\,dt + dW_t\Big).$$

- Under $\mathbb{P}$ the discounted stock drifts up at the excess return $\mu - r$. It is not a fair game: investors are paid to hold risk.

The market price of risk

- $\lambda = \dfrac{\mu - r}{\sigma}$ is the *market price of risk*: excess return per unit of volatility, the Sharpe ratio of the stock.
- It is exactly the drift that must be removed to make the discounted stock a fair game.
- The expected return μ enters only through this one number, and the pricing formulas do not contain it.

Removing the risk premium

Proposition 4.2 (The risk-neutral measure)

Define Q by the Girsanov density $dQ/d\mathbb{P} = e^{-\lambda W_T - \lambda^2 T/2}$ with $\lambda = (\mu - r)/\sigma$. Then $\widetilde{W}_t = W_t + \lambda t$ is a Q-Brownian motion and, under Q,

$$d\widetilde{S}_t = \sigma \widetilde{S}_t \, d\widetilde{W}_t, \qquad dS_t = rS_t \, dt + \sigma S_t \, d\widetilde{W}_t.$$

The discounted stock is a Q-martingale, and Q is the only EMM.

- One reweighting of scenarios removes exactly λ of drift per unit time. Nothing else changes, and nothing else can. A change of measure can only move the drift of W (§3.7, plus one import in the proof).
- μ is gone, absorbed into the measure.

The Black–Scholes PDE

The PDE from no arbitrage: part (a)

Theorem 4.3 (The Black–Scholes PDE), part (a)

Let $C(t,x)$ be continuous on $[0,T]\times(0,\infty)$ with C_t, C_x, C_{xx} continuous on $[0,T)\times(0,\infty)$. Suppose a self-financing portfolio has value $V_t = C(t,S_t)$. Then its share holding is $a_t = C_x(t,S_t)$, and C solves

$$C_t + rx\,C_x + \tfrac{1}{2}\sigma^2 x^2\,C_{xx} = rC \qquad \text{on } [0,T)\times(0,\infty).$$

- If a self-financing portfolio tracks $C(t,S_t)$, then C solves this equation. The equation does not contain μ.
- Proof idea: match the dW terms of Itô's dC with a self-financing portfolio to get $a_t = C_x$, then match the dt terms.

The PDE from no arbitrage: part (b)

Theorem 4.3 (The Black–Scholes PDE), part (b)

Conversely, suppose C solves the equation with $C(T, x) = h(x)$. Then the portfolio with $a_t = C_x(t, S_t)$ shares and $b_t = \big(C(t, S_t) - S_t C_x(t, S_t)\big)/B_t$ units of the account is self-financing, has value $C(t, S_t)$, and ends at $h(S_T)$. So $C(t, S_t)$ is the no-arbitrage price of the claim $h(S_T)$.

- A solution of the equation is a price, because it comes with its own reproducing portfolio.
- Proof idea: substitute the PDE into Itô's drift, and dC becomes $C_x\,dS + r(C - SC_x)\,dt$, which is $a\,dS + b\,dB$.

The classical reading: a riskless book

- Be long one option and short C_x shares. By Itô the book's dW terms cancel, so it is riskless over the next instant.
- A riskless book must earn r. Writing that down is the PDE.
- This book is not self-financing as written, because C_x moves. The careful version matches the dW and dt terms of dC with those of a self-financing portfolio. See Steps 2–4 of the proof.

The PDE from the CAPM

Proposition 4.4 (The CAPM route to the PDE)

Let the claim's price be $C(t, S_t)$ with C as in Theorem 4.3. Assume the continuous-time CAPM: there is a market portfolio M with expected return μ_M such that every traded asset X satisfies

$$\frac{\mathbb{E}^{\mathbb{P}}[dX_t \mid \mathcal{F}_t]}{X_t} = \big(r + \beta_X(\mu_M - r)\big)dt, \qquad \beta_X = \frac{\mathrm{Cov}(dX/X,\, dM/M \mid \mathcal{F}_t)}{\mathrm{Var}(dM/M \mid \mathcal{F}_t)}.$$

Then C solves the Black–Scholes PDE.

- The option's noise is (SC_x/C) times the stock's, so $\beta_C = (SC_x/C)\beta_S$; then μ cancels against Itô's drift. An equilibrium argument, not a no-arbitrage one.

Feynman–Kac: the Price as an Expectation

From the PDE to the expectation

Theorem 4.5 (Feynman–Kac and the price), part (a)

Let h be continuous with $\mathbb{E}^{Q}|h(S_T)| < \infty$, and let C be as in Theorem 4.3. Suppose C solves the Black–Scholes PDE on $[0, T) \times (0, \infty)$ with $C(T, x) = h(x)$, and $\mathbb{E}^{Q} \int_0^T \sigma^2 S_s^2 C_x^2(s, S_s)\, ds < \infty$. Then for every $t \le T$,

$$C(t, S_t) = e^{-r(T-t)}\, \mathbb{E}^{Q}\left[h(S_T) \,\middle|\, \mathcal{F}_t\right].$$

- No arbitrage forces the price to solve the PDE (Theorem 4.3(a)). Part (a) then makes the price a discounted Q-expectation.
- Proof idea: by the PDE, $e^{-rt}C(t, S_t)$ has no dt term under Q, so it is a martingale.

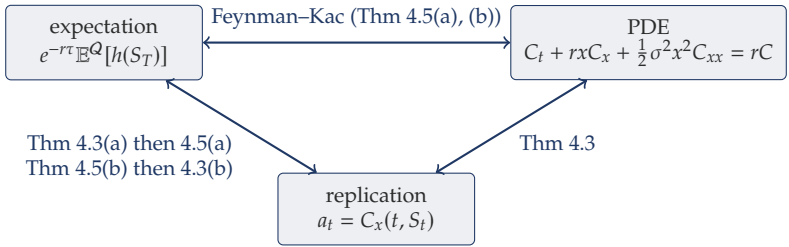
From the expectation to the PDE

Theorem 4.5 (Feynman–Kac and the price), part (b)

Let h be continuous with $\mathbb{E}^Q|h(S_T)| < \infty$, and define $C(t,x) = e^{-r(T-t)}\,\mathbb{E}^Q\left[h(S_T) \,\middle|\, S_t = x\right]$. Assume C satisfies hypotheses (iii)–(v) of the Feynman–Kac theorem of §3.8 for S under Q, on $(0,\infty)$. Then C solves the Black–Scholes PDE on $[0,T)\times(0,\infty)$ with $C(T,x) = h(x)$, and $C(t,S_t)$ is the no-arbitrage price of $h(S_T)$.

- Under Q the stock has drift rx and diffusion σx (Proposition 4.2), the PDE's coefficients. So Feynman–Kac applies, and Theorem 4.3(b) prices the solution.
- Part (a) assumes a PDE solution exists; part (b) supplies one.

One price, three descriptions



- Girsanov (§3.7) builds Q in the expectation box. Feynman–Kac runs both ways, (a) and (b). Each description feeds its own method in Section 5, and hedging.
- Backward induction on the tree is the discrete Feynman–Kac; the share-and-bond portfolio is the discrete C_x hedge.

The Black–Scholes Formula

The Black–Scholes formula

Theorem 4.6 (Black–Scholes formula)

With $\tau = T - t$, the European call is worth

$$C(S,\tau) = S\,\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \qquad d_{1,2} = \frac{\ln(S/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$.

- The proof computes the expectation of Theorem 4.5(a) in three moves: lognormal S_T; a stock leg and a cash leg; a completed square.
- Both Φ's are probabilities of finishing in the money, under two *different* measures. §4.6 identifies them.

Put–call parity

- A long call and a short put at the same strike pay $S_T - K$ together. That is a forward, worth $S - Ke^{-r\tau}$ by Section 1 replication.
- So

$$C - P = S - Ke^{-r\tau}.$$

- Model-free: it needs only the law of one price, not lognormality.

Example 4.7

Example 4.7

$S = K = 100$, $r = 0$, $\sigma = 0.2$, $\tau = 1$, using $\Phi(0.1) = 0.5398$. Compute d_1, d_2, the call price, and the put price via parity.

The Share Measure and the Meaning of $\Phi(d_1)$

Probabilities of what, under which measure?

- The formula reads: S times a probability, minus $Ke^{-r\tau}$ times a probability. Which probabilities, under which measures?
- The second one is in Step 1 of the proof of Theorem 4.6:

$$Q(S_T > K) = Q(Z > -d_2) = \Phi(d_2).$$

  So $\Phi(d_2)$ is the risk-neutral probability of exercise.
- The first one needs a second measure: one that counts value in shares rather than in units of the money market. We call it the *share measure* Q^S.

The share measure

Proposition 4.8 (The share measure)

Define Q^S by $\dfrac{dQ^S}{dQ} = \dfrac{S_T}{S_0 e^{rT}} = e^{\sigma \widetilde{W}_T - \sigma^2 T/2}$. Then $\widehat{W}_t = \widetilde{W}_t - \sigma t$ is a Q^S-Brownian motion, $Q^S(S_T > K) = \Phi(d_1)$, and

$$C = S\,Q^S(S_T > K) - Ke^{-r\tau}\,Q(S_T > K).$$

- The density is the Girsanov density of §3.7 with $\lambda = -\sigma$. Under Q^S the growth rate of the stock gains a full σ^2.
- If value is counted in shares instead of dollars, the probability of finishing in the money rises from $\Phi(d_2)$ to $\Phi(d_1)$.

What a numeraire is

- A numeraire is the asset in whose units value is counted.
- Under Q the numeraire is the money market: every price divided by B_t is a martingale.
- Under Q^S the numeraire is the stock: every price divided by S_t is a martingale. That is the change-of-numeraire theorem (Björk); B_t/S_t takes one line.
- The density $S_T/(S_0 e^{rT})$ is the stock's terminal value over its forward value. Q^S gives more weight to the scenarios in which the stock did well, in proportion to how well it did.

Two legs, two currencies

- The call receives one share and pays K dollars, both only if $S_T > K$.
- The share leg is counted in shares: one unit of the numeraire S with probability $Q^S(S_T > K)$. Worth $S \cdot \Phi(d_1)$.
- The cash leg is counted in dollars: K dollars at T with probability $Q(S_T > K)$. Worth $Ke^{-r\tau}\Phi(d_2)$.
- Each leg uses the exercise probability of its own currency. The formula is the difference.

Example 4.9

Example 4.9

A *digital (cash-or-nothing) call* pays 1 if $S_T > K$ and 0 otherwise. Price it in the Black–Scholes market. Evaluate it on the ATM numbers of Example 4.7, and state the digital call–put parity.

The Greeks

One identity does most of the work

Lemma 4.10

With d_1, d_2 as in Theorem 4.6 and φ the standard normal density,

$$S\,\varphi(d_1) = Ke^{-r\tau}\,\varphi(d_2).$$

- The two density terms that appear whenever the formula is differentiated are equal. So they cancel in pairs, or merge.
- Proof: $d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2) = 2(\ln(S/K) + r\tau)$, then exponentiate.

The Greeks of the call

Theorem 4.11 (Greeks of the call)

$$\Delta = \frac{\partial C}{\partial S} = \Phi(d_1), \qquad \Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\varphi(d_1)}{S\sigma\sqrt{\tau}}, \qquad \Theta = \frac{\partial C}{\partial t} = -\frac{S\varphi(d_1)\,\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2),$$

$$\text{vega} = \frac{\partial C}{\partial \sigma} = S\varphi(d_1)\sqrt{\tau}, \qquad \rho = \frac{\partial C}{\partial r} = K\tau e^{-r\tau}\Phi(d_2).$$

- $\Delta = \Phi(d_1)$ is the share holding C_x of Theorem 4.3(b).

Two relations among the Greeks

Proposition 4.12 (Two relations among the Greeks)

$$\text{(a)} \quad \Theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = rC, \qquad \text{(b)} \quad \text{vega} = \sigma\tau S^2\,\Gamma.$$

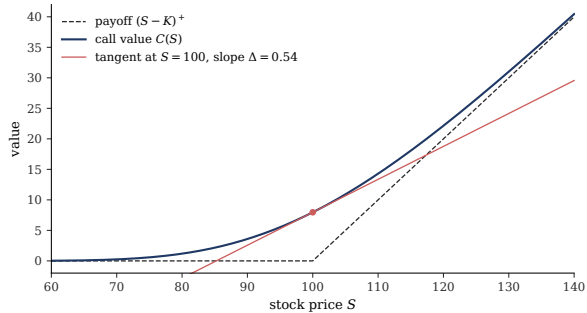
- (a) is the Black–Scholes PDE written in Greek letters.
- (b): sensitivity to volatility and convexity in the stock are the same quantity up to the factor $\sigma\tau S^2$.

Example 4.13

Example 4.13

On the ATM numbers of Example 4.7, with $\varphi(0.1) = 0.3970$ given, compute Δ, Γ, Θ and vega. Then verify both relations of Proposition 4.12.

The value curve



- The payoff has a kink at K; the value curve is smooth. Its slope at $S = 100$ is $\Delta = 0.54$, and its curvature is Γ.

Dividends and Options on Futures

A continuous dividend yield: dynamics and PDE

Theorem 4.14 (Continuous dividend yield), parts (a) and (b)

Let the stock pay dividends at rate q, with $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ under $\mathbb{P}$. Then:

(a) Under $\boldsymbol{Q}$, defined by the Girsanov density with $\lambda = (\mu + q - r)/\sigma$, $dS_t = (r-q)S_t\,dt + \sigma S_t\,d\widetilde{W}_t$, and $S_t e^{qt}/B_t$ is a $\boldsymbol{Q}$-martingale.

(b) If a self-financing portfolio has value $C(t, S_t)$, with C as in Theorem 4.3, then C solves $C_t + (r-q)x\,C_x + \frac{1}{2}\sigma^2 x^2\,C_{xx} = rC$.

– The traded asset is the reinvested stock $S_t e^{qt}$. The dividend lowers the risk-neutral growth rate of the price from r to $r - q$; the holder still earns r in total.

A continuous dividend yield: the formula

Theorem 4.14 (Continuous dividend yield), part (c)

(c) The European call is worth

$$C = Se^{-q\tau}\Phi(d_1) - Ke^{-r\tau}\Phi(d_2), \qquad d_{1,2} = \frac{\ln(S/K) + \left(r - q \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

- Every S becomes $Se^{-q\tau}$: the share received at exercise has missed the dividends paid before T. The proof of Theorem 4.6 goes through with that substitution.

Options on futures: dynamics and PDE

Theorem 4.15 (Black's formula for options on futures), parts (a) and (b)

Let $F_t = S_t e^{(r-q)(T^*-t)}$ be the futures price on the stock of Theorem 4.14. Consider a European call on the futures with strike K and maturity $T \le T^*$.

(a) Under Q, $dF_t = \sigma F_t \, d\widetilde{W}_t$: the futures price is a Q-martingale.

(b) If a self-financing portfolio of futures and the account has value $C(t, F_t)$, with C as in Theorem 4.3, then C solves $C_t + \frac{1}{2}\sigma^2 x^2 C_{xx} = rC$.

– A futures position costs nothing, so it earns nothing on average under Q, and the PDE has no first-order term.

Options on futures: the formula

Theorem 4.15 (Black's formula for options on futures), part (c)

(c) The call is worth

$$C = e^{-r\tau}\left[F\,\Phi(d_1) - K\,\Phi(d_2)\right], \qquad d_{1,2} = \frac{\ln(F/K) \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}.$$

- This is Theorem 4.14(c) with $q = r$ and $S = F$. A futures contract behaves like a stock whose dividend yield equals the riskless rate. Black (1976).

Example 4.16

Example 4.16

A European call on a futures contract has $F = K = 100$, $\sigma = 0.2$, $\tau = 1$ and $r = 5\%$. Use $\Phi(0.1) = 0.5398$ and $e^{-0.05} = 0.9512$. Compute its price.