Section 4 The Black–Scholes World

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Overview

This section builds the Black–Scholes theory from the tools of Section 3. §4.1 states the assumptions and explains the ideas in words. §4.2 changes measure once, with Girsanov, and finds the stock's dynamics under the risk-neutral measure. §4.3 derives the Black–Scholes partial differential equation twice, from a self-financing portfolio and from the CAPM. §4.4 applies Feynman–Kac and writes the price as a discounted expectation. §4.5 computes that expectation and obtains the Black–Scholes formula. §4.6 changes measure a second time and explains what $\Phi(d_1)$ and $\Phi(d_2)$ mean. §4.7 derives the Greeks. §4.8 extends the formula to a stock that pays dividends and to options on futures. The stochastic-calculus tools used here were built in Section 3, and each use names the subsection it comes from.

4.1 The Black–Scholes Assumptions and Ideas

This subsection fixes the assumptions and explains the ideas in words. The computations start in §4.2.

The Black–Scholes model makes five assumptions.

A1. Constant volatility.
The volatility σ of the stock is a known constant. So S_T is lognormal, and there is one number to calibrate.

A2. One constant rate.
Borrowing and lending happen at the same constant rate r. So there is one discount factor, $e^{-r\tau}$, where $\tau = T - t$ is the time left to the option's maturity T.

A3. No dividends.
The stock pays nothing before T. §4.8 removes this assumption.

A4. Frictionless continuous trading.
There are no transaction costs and no taxes. Any fraction of a share can be traded at any instant. This is what makes exact replication possible.

A5. Continuous paths.
The stock follows a geometric Brownian motion. Its path never jumps, so a hedge can be adjusted through every move.

Under these assumptions the theory rests on five ideas.

First, the option is a redundant asset. Its payoff $h(S_T)$ can be reproduced by a position in the stock and the money market. The position is adjusted continuously, and no money is added or withdrawn along the way. §4.3 constructs this position.

Second, the price is fixed by no arbitrage alone. Suppose the option traded above the cost of the reproducing position. Then one would sell the option and buy the position, and keep the

difference at no risk. So the option must cost exactly what the position costs. No risk preference enters. This is the logic of Section 1 in continuous time.

Third, the price is a function $C(t, S)$ of two variables. By the solution of §3.4,

$$S_T = S_t\, e^{(\mu-\sigma^2/2)(T-t)+\sigma(W_T-W_t)},$$

and the increment $W_T - W_t$ is independent of everything known at t (§3.1). So the date t and the current price S_t are all the option's value can depend on. Everything in this section is a statement about this function.

Fourth, μ plays no role and σ does. Two investors may disagree about the expected return of the stock but agree on its volatility. Then they must agree on every option price. §4.2 shows where μ disappears.

Fifth, one price has three descriptions. The function C solves a partial differential equation (§4.3). It is a discounted expectation under a special probability measure (§4.4). It is the value of the reproducing position (§4.3 again). The section walks through the three in that order and shows that they agree.

We now fix the market.

Definition 4.1 (Black–Scholes market).
Under the physical measure $\mathbb{P}$, the market consists of

a stock

following geometric Brownian motion, $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$, with volatility $\sigma > 0$. By §3.4 its solution is $S_t = S_0 e^{(\mu-\sigma^2/2)t+\sigma W_t}$;

a money market account

$B_t = e^{rt}$, growing at the constant riskless rate r, so $dB_t = rB_t\, dt$.

A portfolio holds a_t shares and b_t units of the account, both adapted. Its value is $V_t = a_t S_t + b_t B_t$. The portfolio is self-financing *if $dV_t = a_t\, dS_t + b_t\, dB_t$.*

In a self-financing portfolio every change in value comes from price moves. None comes from money added or withdrawn. This is Section 2's self-financing definition, with the adjustments now continuous.

Arbitrage and the equivalent martingale measure carry over from Section 2, with the tree replaced by the Brownian filtration. An arbitrage is a self-financing strategy with $V_0 \le 0$ and $V_T \ge 0$, with at least one strict inequality. An equivalent martingale measure (EMM) is an equivalent measure under which discounted prices are martingales. Continuous time adds one technical condition on strategies, called admissibility. It rules out doubling strategies and changes no price. We acknowledge it and set it aside.

4.2 The Stock under the Risk-Neutral Measure

This subsection performs the calculation that every later section reuses. One application of Girsanov removes the risk premium of the stock.

Discount the stock: $\widetilde{S}_t = S_t/B_t$. We compute its dynamics in full, because the computation is a template used throughout the course. Since $B_t = e^{rt}$ is deterministic, $d(1/B_t) = -r(1/B_t)\, dt$,

with no dW term. The Itô product rule of §3.5 reads $d(XY) = X\,dY + Y\,dX + d[X,Y]$. Here the bracket term is empty, because $1/B$ carries no noise. So

$$d\widetilde{S}_t = \frac{1}{B_t}\,dS_t + S_t\,d\left(\frac{1}{B_t}\right) = \frac{S_t}{B_t}\big(\mu\,dt + \sigma\,dW_t\big) - r\,\frac{S_t}{B_t}\,dt = \widetilde{S}_t\big((\mu - r)\,dt + \sigma\,dW_t\big).$$

Factor out the volatility:

$$d\widetilde{S}_t = \sigma\widetilde{S}_t\Big(\underbrace{\tfrac{\mu-r}{\sigma}}_{\lambda}\,dt + dW_t\Big).$$

Under $\mathbb{P}$ the discounted stock drifts at the excess return $\mu - r$. It is not a fair game, because investors are paid to hold risk. The number

$$\lambda \;=\; \frac{\mu - r}{\sigma}$$

is the *market price of risk*: the excess return per unit of volatility. It is the Sharpe ratio of the stock. It is also exactly the drift that must be removed to make the discounted stock a fair game.

Proposition 4.2 (The risk-neutral measure)**.**
Define $\mathbf{Q}$ by the Girsanov density $d\mathbf{Q}/d\mathbb{P} = e^{-\lambda W_T - \lambda^2 T/2}$ with $\lambda = (\mu - r)/\sigma$. Then $\widetilde{W}_t = W_t + \lambda t$ is a $\mathbf{Q}$-Brownian motion, and under $\mathbf{Q}$

$$d\widetilde{S}_t = \sigma\widetilde{S}_t\,d\widetilde{W}_t, \qquad dS_t = rS_t\,dt + \sigma S_t\,d\widetilde{W}_t.$$

So the discounted stock is a $\mathbf{Q}$-martingale, and under $\mathbf{Q}$ the stock earns the riskless rate. Moreover $\mathbf{Q}$ is the only EMM in this market.

There is exactly one way to reweight scenarios so that holding the stock is not rewarded beyond the riskless rate. The reweighting removes λ of drift per unit time. Nothing else changes. A change of measure can only move the drift of W. The volatility is measure-invariant (§3.7). The proof imports the rest of this claim.

Proof.

□

Note what happened to μ. It is gone, absorbed into the measure. No pricing formula in this section will contain it. This is Section 1's lesson again, now in continuous time: physical probabilities drop out of relative pricing.

4.3 The Black–Scholes PDE

This subsection derives the equation that the price function $C(t,x)$ satisfies. It does so twice. The first derivation uses no arbitrage. The second uses the CAPM. Both give the same equation, and in both the expected return μ cancels.

Throughout, $C(t,x)$ is a function of the date t and the stock price x. Its partial derivatives are C_t, C_x and C_{xx}. We write $C(t,S_t)$ for the function evaluated along the stock path.

Theorem 4.3 (The Black–Scholes PDE)**.**
Let $C(t,x)$ be continuous on $[0,T]\times(0,\infty)$, with C_t, C_x, C_{xx} continuous on $[0,T)\times(0,\infty)$.

(a) Suppose a self-financing portfolio has value $V_t = C(t,S_t)$. Then its share holding is $a_t = C_x(t,S_t)$, and C solves

$$C_t + rx\,C_x + \tfrac{1}{2}\sigma^2x^2\,C_{xx} = rC \qquad \text{on } [0,T)\times(0,\infty).$$

(b) Conversely, suppose C solves this equation with $C(T,x)=h(x)$. Then the portfolio with $a_t = C_x(t,S_t)$ shares and $b_t = \big(C(t,S_t) - S_tC_x(t,S_t)\big)/B_t$ units of the account is self-financing, has value $C(t,S_t)$, and ends at $h(S_T)$. So $C(t,S_t)$ is the no-arbitrage price of the claim $h(S_T)$.

Part (a) says: if a self-financing portfolio tracks $C(t,S_t)$, then C solves this equation. By the law of one price, the price of a replicable claim is the value of its replicating portfolio. So the price solves the equation. Part (b) says: a solution of the equation is a price, because it comes with its own reproducing portfolio. The equation does not contain μ. The classical reading of (a) is a book that is long one option and short C_x shares. Its dW terms cancel, so the book is riskless over the next instant. A riskless book must earn r. This book is not self-financing as written, because C_x moves. Steps 2–4 of the proof are the careful version of this reading.

Proof.

□

The second derivation replaces the no-arbitrage argument by an equilibrium model of expected returns. The CAPM says that the expected excess return on any asset is proportional to its beta against the market portfolio. We take the continuous-time form of this statement as an assumption.

Proposition 4.4 (The CAPM route to the PDE)**.**
Let $C(t,x)$ be as in Theorem 4.3, and let the claim's price be $C(t,S_t)$. Assume the continuous-time CAPM: there is a market portfolio with price M_t and expected return μ_M such that every traded asset with price X_t satisfies

$$\frac{\mathbb{E}^{\mathbb{P}}[dX_t \mid \mathcal{F}_t]}{X_t} = \big(r + \beta_X(\mu_M - r)\big)\,dt, \qquad \beta_X = \frac{\operatorname{Cov}(dX_t/X_t,\, dM_t/M_t \mid \mathcal{F}_t)}{\operatorname{Var}(dM_t/M_t \mid \mathcal{F}_t)}.$$

Then C solves the Black–Scholes PDE of Theorem 4.3.

The CAPM is an equilibrium argument: it assumes that investors hold the market portfolio. Theorem 4.3 assumes only that there is no arbitrage. Both give the same equation because μ cancels either way. This second derivation is the alternative derivation in Black and Scholes (1973), *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy.

Proof.

□

4.4 Feynman–Kac: the Price as an Expectation

This subsection turns the PDE into a formula. Theorem 4.3 says what the price must satisfy: a partial differential equation with terminal value h. The Feynman–Kac theorem of §3.8 links such an equation to a discounted conditional expectation of a diffusion. The link runs in both directions. The expectation solves the equation. A solution of the equation, under an integrability condition, is the expectation. The coefficients of the equation are the diffusion's drift and half its squared volatility. The Black–Scholes PDE has drift coefficient rx and diffusion coefficient σx. By Proposition 4.2, those are exactly the coefficients of the stock under Q. So the expectation is taken under Q, and that is why Q was built first.

Theorem 4.5 (Feynman–Kac and the price)**.**
Let h be continuous with $\mathbb{E}^Q|h(S_T)| < \infty$.

(a) (From the PDE to the expectation.) Let $C(t,x)$ be as in Theorem 4.3. Suppose C solves the Black–Scholes PDE on $[0,T)\times(0,\infty)$ with $C(T,x) = h(x)$, and $\mathbb{E}^Q\int_0^T \sigma^2 S_s^2\, C_x^2(s,S_s)\,ds < \infty$. Then for every $t \le T$,

$$C(t,S_t) \;=\; e^{-r(T-t)}\,\mathbb{E}^Q\big[\,h(S_T)\,\big|\,\mathcal{F}_t\,\big].$$

(b) (From the expectation to the PDE.) Define

$$C(t,x) \;=\; e^{-r(T-t)}\,\mathbb{E}^Q\big[\,h(S_T)\,\big|\,S_t = x\,\big],$$

and assume that C satisfies hypotheses (iii)–(v) of the Feynman–Kac theorem of §3.8, for S under Q and on the state space $(0,\infty)$. Then C solves the Black–Scholes PDE on $[0,T)\times(0,\infty)$ with $C(T,x) = h(x)$, and $C(t,S_t)$ is the no-arbitrage price of the claim $h(S_T)$.

Part (a) is the direction this section runs in. No arbitrage forces the price to solve the PDE (Theorem 4.3(a)). Part (a) then writes every such solution, under its integrability condition, as a discounted expectation under Q. So the price of a claim is its discounted expectation under Q. This is the same rule as in the one-period and the tree models of Sections 1 and 2. Here it is derived from the tools of Section 3.

Part (b) runs the other way: from the expectation to the PDE, and then Theorem 4.3(b) makes it a price. Part (a) assumes that a PDE solution exists. Part (b) supplies one. For the call, §4.5 computes the expectation in closed form, and §4.7 checks that the closed form solves the PDE. Part (a) is the converse half of the Feynman–Kac theorem of §3.8, proved here for this market. Part (b) is its direct half. Like §3.8, it assumes the smoothness of C rather than deriving it.

Proof.

□

Figure 1 collects what has been established. The same number can be read three ways, and each reading has its own computational method.

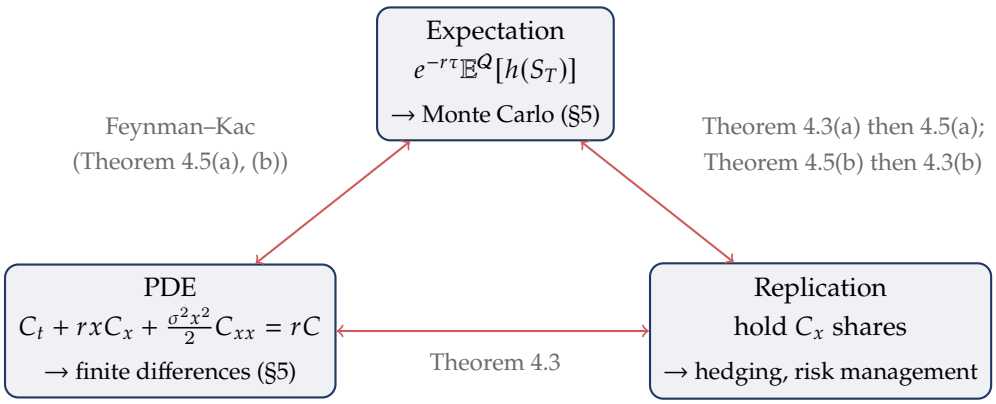


Figure 1: One price, three descriptions. Girsanov (§3.7) builds the measure Q that sits at the top. Feynman–Kac links the top to the PDE in both directions. Theorem 4.5(a) goes from the PDE to the expectation. Part (b) goes back. Theorem 4.3 turns a PDE solution into a reproducing portfolio. Each description feeds its own computational method in Section 5.

The triangle also answers a question left open in Section 2. Backward induction on the tree computes "value equals discounted expected value one step ahead". That is the discrete Feynman–Kac. The share-and-bond portfolio at each node is the discrete C_x hedge. Section 5 derives that portfolio. The binomial model is the same theory in discrete time.

4.5 The Black–Scholes Formula

This subsection computes the expectation of Theorem 4.5(a) for the European call, $h(S_T) = (S_T - K)^+$. The computation conditions on $S_t = S$. That is enough. Under Q, $S_T = S_t\, e^{(r-\sigma^2/2)\tau+\sigma(\widetilde{W}_T-\widetilde{W}_t)}$, and the increment $\widetilde{W}_T - \widetilde{W}_t$ is independent of $\mathcal{F}_t$ (§3.1). So the conditional expectation given $\mathcal{F}_t$ is a function of S_t alone.

Theorem 4.6 (Black–Scholes formula)**.**
The value at time t of a European call with strike K and maturity T, with $\tau = T - t$, is

$$C(S,\tau) \;=\; S\,\Phi(d_1) \;-\; Ke^{-r\tau}\,\Phi(d_2), \qquad d_{1,2} \;=\; \frac{\ln(S/K) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$

where Φ is the standard normal distribution function and $d_2 = d_1 - \sigma\sqrt{\tau}$.

The call price is the stock times one number between 0 and 1, minus the discounted strike times another. Both numbers are probabilities of finishing in the money, under two different measures. §4.6 identifies them.

Proof.

□

One remark before the numbers. A long call and a short put at the same strike pay $S_T - K$ together. That is a forward, and its value is $S - Ke^{-r\tau}$ by Section 1 replication: hold the stock and borrow the discounted strike. So

$$C - P = S - Ke^{-r\tau}.$$

This is *put–call parity*. It is model-free: it needs only the law of one price, not lognormality.

Example 4.7. Take $S = K = 100$, $r = 0$, $\sigma = 0.2$, $\tau = 1$, and use $\Phi(0.1) = 0.5398$. Compute d_1, d_2, the call price, and the put price via parity.

4.6 The Share Measure and the Meaning of $\Phi(d_1)$

The formula reads "S times a probability minus $Ke^{-r\tau}$ times a probability". This subsection says which probabilities, under which measures, and why.

The second probability is already in Step 1 of the proof of Theorem 4.6:

$$Q(S_T > K) = Q(Z > -d_2) = \Phi(d_2).$$

So $\Phi(d_2)$ is the risk-neutral probability of exercise. The first probability needs a second measure. That measure uses the stock, not the money market, as the unit in which value is counted. We call it the *share measure* Q^S. Step 3 of the proof of Theorem 4.6 completed a square. In the language of measures, that square is this change of measure. Proposition 4.8 makes the statement exact. It also gives a second way to compute $\Phi(d_1)$, without completing any square.

Proposition 4.8 (The share measure).
Define Q^S by the Radon–Nikodym derivative

$$\frac{dQ^S}{dQ} = \frac{S_T}{S_0\, e^{rT}} = e^{\sigma\widetilde{W}_T - \sigma^2 T/2}.$$

Then under Q^S the process $\widehat{W}_t = \widetilde{W}_t - \sigma t$ is a Brownian motion, and

$$Q^S(S_T > K) = \Phi(d_1), \qquad \text{so} \qquad C = S\,Q^S(S_T > K) - Ke^{-r\tau}\,Q(S_T > K).$$

If value is counted in shares instead of dollars, the probability of finishing in the money rises from $\Phi(d_2)$ to $\Phi(d_1)$. The two paragraphs after the proof explain why.

Proof.

□

What a numeraire is. A numeraire is the asset in whose units value is counted. Under Q the numeraire is the money market. The price of every asset divided by B_t is a martingale. Proposition 4.2 shows this for the stock. Under Q^S the numeraire is the stock. The price of every asset divided by S_t is a martingale. For B_t/S_t the check is one line. By Step 2 of the proof, $S_t = S_0 e^{(r+\sigma^2/2)t+\sigma\widehat{W}_t}$, so $B_t/S_t = (1/S_0)\, e^{-\sigma\widehat{W}_t-\sigma^2 t/2}$, the exponential martingale of §3.4 under Q^S. The statement for every traded asset is the change-of-numeraire theorem. We state it without proof (Björk, *Arbitrage Theory in Continuous Time*). The density $S_T/(S_0 e^{rT})$ is the terminal value of the stock divided by its forward value. So Q^S gives more weight to the scenarios in which the stock did well. The extra weight is in proportion to how well it did.

Two legs, two currencies. The call receives one share and pays K dollars, both only if $S_T > K$. The share leg is counted in shares. It delivers one unit of the numeraire S with probability $Q^S(S_T > K)$, so it is worth $S \cdot \Phi(d_1)$. The cash leg is counted in dollars. It delivers K dollars at T with probability $Q(S_T > K)$, so it is worth $Ke^{-r\tau}\Phi(d_2)$. Each leg uses the exercise probability of its own currency. The formula is the difference.

Example 4.9. A *digital (cash-or-nothing) call* pays 1 if $S_T > K$ and 0 otherwise. Price it in the Black–Scholes market. Evaluate it on the ATM numbers of Example 4.7, and state the digital call–put parity.

4.7 The Greeks

The Greeks are the partial derivatives of the pricing function. This subsection derives them from Theorem 4.6 and proves two relations among them. One identity does most of the work.

Lemma 4.10.
With d_1, d_2 as in Theorem 4.6 and φ the standard normal density,

$$S\,\varphi(d_1) \;=\; Ke^{-r\tau}\,\varphi(d_2).$$

The two density terms that appear whenever the formula is differentiated are equal. So they cancel in pairs, or merge.

Proof.

□

Theorem 4.11 (Greeks of the call).
For the call of Theorem 4.6,

$$\Delta = \frac{\partial C}{\partial S} = \Phi(d_1), \qquad \Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\varphi(d_1)}{S\sigma\sqrt{\tau}}, \qquad \Theta = \frac{\partial C}{\partial t} = -\frac{S\varphi(d_1)\,\sigma}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2),$$

$$\text{vega} = \frac{\partial C}{\partial \sigma} = S\varphi(d_1)\sqrt{\tau}, \qquad \rho = \frac{\partial C}{\partial r} = K\tau e^{-r\tau}\Phi(d_2).$$

$\Delta = \Phi(d_1)$ is the share holding C_x of Theorem 4.3(b). The hedge ratio of the PDE proof now has a closed form. Γ and vega are positive: the call is convex in the stock and gains from volatility. Θ is negative when $r \geq 0$: the call loses value as time passes with nothing else moving.

Proof.

□

Proposition 4.12 (Two relations among the Greeks).
For the call of Theorem 4.6,

$$\text{(a)} \quad \Theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = rC, \qquad \text{(b)} \quad \text{vega} = \sigma\tau S^2\,\Gamma.$$

Relation (a) is the Black–Scholes PDE written in Greek letters. Relation (b) says that vega and Γ are the same quantity up to the factor $\sigma\tau S^2$.

Proof.

□

Relation (a) also shows that the closed form of Theorem 4.6 solves the Black–Scholes PDE for $t < T$. The closed form is continuous on $[0, T] \times (0, \infty)$ with $C(T, x) = (x - K)^+$. So Theorem 4.3(b) makes it the no-arbitrage price directly. This closes the circle of §4.4 for the call. A PDE solution exists, namely the closed form. By the computation of §4.5, it is the expectation of Theorem 4.5(a). The hypotheses (iii)–(v) of Theorem 4.5(b) are not needed.

Example 4.13. On the ATM numbers of Example 4.7, with $\varphi(0.1) = 0.3970$ given, compute Δ, Γ, Θ and vega. Then verify both relations of Proposition 4.12.

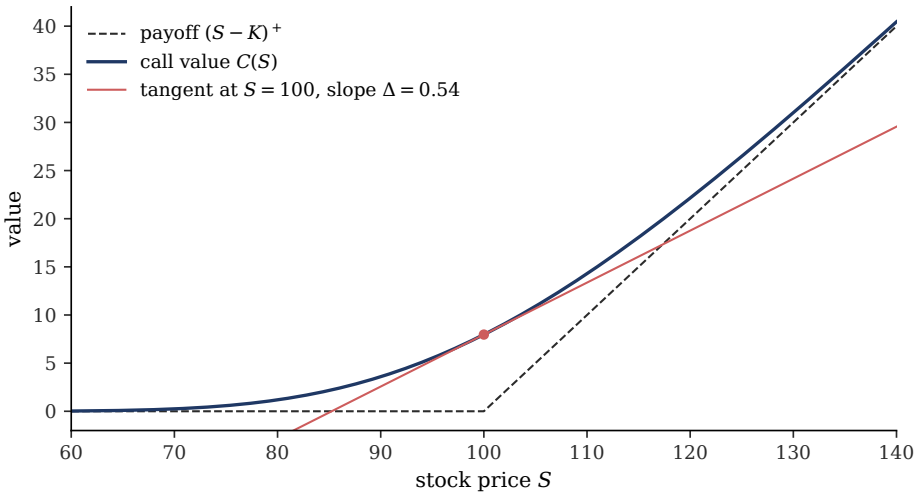


Figure 2: The call value against the stock price, on the ATM parameters of Example 4.7. The payoff (dashed) has a kink at $K = 100$. The value curve (solid) is smooth, because the expectation of Theorem 4.5(a) averages the kink over the lognormal distribution of S_T. Its slope at $S = 100$ is $\Delta = 0.54$ (the tangent line), and its curvature is Γ. Produced by `figures/bs_value_curve.py`.

4.8 Dividends and Options on Futures

This subsection removes assumption A3 and then prices an option written on a futures contract. Both results follow from the proofs already given, with one change each. The proofs state exactly which change.

Suppose the stock pays dividends continuously at the constant rate q. A holder of one share receives $qS_t\,dt$ in cash over $[t, t+dt]$. So the self-financing condition of Definition 4.1 must include this cash. A portfolio of a_t shares and b_t units of the account is self-financing if

$$dV_t = a_t\,(dS_t + qS_t\,dt) + b_t\,dB_t.$$

Theorem 4.14 (Continuous dividend yield).
Let the stock pay dividends at rate q, with price $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$ under $\mathbb{P}$. Then:

(a) Under $\mathbf{Q}$, defined by the Girsanov density with $\lambda = (\mu + q - r)/\sigma$, the stock follows $dS_t = (r-q)S_t\,dt + \sigma S_t\,d\widetilde{W}_t$, and $S_t e^{qt}/B_t$ is a $\mathbf{Q}$-martingale.

(b) If a self-financing portfolio has value $C(t, S_t)$, with C as in Theorem 4.3, then C solves $C_t + (r-q)x\,C_x + \frac{1}{2}\sigma^2 x^2\,C_{xx} = rC$.

(c) The European call is worth

$$C = Se^{-q\tau}\,\Phi(d_1) - Ke^{-r\tau}\,\Phi(d_2), \qquad d_{1,2} = \frac{\ln(S/K) + \left(r - q \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

The dividend lowers the risk-neutral growth rate of the price from r to $r - q$. The holder of the stock still earns r in total, because the dividends make up the difference. In the formula every S becomes $Se^{-q\tau}$: the share you receive at exercise has missed the dividends paid before T.

Proof.

□

Now let F_t be the futures price at t for delivery of the stock at $T^* \geq T$. With a constant rate, futures and forward prices coincide. We import this result from Cox, Ingersoll and Ross (1981), *The relation between forward prices and futures prices*, Journal of Financial Economics. So F_t is the forward price, and Section 1 replication gives it. Buy $e^{-q(T^*-t)}$ shares now and reinvest the dividends. This delivers one share at T^* for a cost of $S_t e^{-q(T^*-t)}$ today. A forward contract costs nothing at t, so its delivery price is this cost compounded at r to T^*:

$$F_t = S_t \, e^{(r-q)(T^*-t)}.$$

A futures position costs nothing to enter. Marking to market resets its value to zero at every instant. Its gains are paid in cash as the futures price moves: a holder of a_t contracts receives $a_t \, dF_t$ over $[t, t+dt]$. So a portfolio of a_t contracts and b_t units of the account has value $V_t = b_t B_t$. It is self-financing if $dV_t = a_t \, dF_t + b_t \, dB_t$.

Theorem 4.15 (Black's formula for options on futures)**.**
Let $F_t = S_t e^{(r-q)(T^*-t)}$ be the futures price on the stock of Theorem 4.14. Consider a European call on the futures contract with strike K and maturity $T \le T^*$. Then:

(a) Under $\mathbf{Q}$, $dF_t = \sigma F_t \, d\widetilde{W}_t$: the futures price is a $\mathbf{Q}$-martingale.

(b) If a self-financing portfolio of futures and the account has value $C(t, F_t)$, with C as in Theorem 4.3, then C solves $C_t + \frac{1}{2}\sigma^2 x^2 C_{xx} = rC$.

(c) The call is worth

$$C = e^{-r\tau}\big[F\,\Phi(d_1) - K\,\Phi(d_2)\big], \qquad d_{1,2} = \frac{\ln(F/K) \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}.$$

A futures position costs nothing, so under $\mathbf{Q}$ it must earn nothing on average. So the futures price has no drift. The PDE has no first-order term for the same reason. The formula is Theorem 4.14(c) with $q = r$ and $S = F$. A futures contract behaves like a stock whose dividend yield equals the riskless rate. Source: Black (1976), *The pricing of commodity contracts*, Journal of Financial Economics.

Proof.

□

Example 4.16. A European call on a futures contract has $F = K = 100$, $\sigma = 0.2$, $\tau = 1$ and $r = 5\%$. Use $\Phi(0.1) = 0.5398$ and $e^{-0.05} = 0.9512$. Compute its price.

Exercises

Exercise 4.1. *Show that $\mathbb{E}^{Q}[S_T] = S_0 e^{rT}$, using the normal moment generating function and the Q-dynamics of Proposition 4.2. Then explain in one sentence why "under Q every asset earns r" is an accounting identity and not an economic forecast.*

Exercise 4.2. *Derive put–call parity $C - P = S - Ke^{-r\tau}$ by exhibiting two portfolios with identical payoffs. Verify it on the numbers of Example 4.7. Which sections' logic does the derivation use, and which does it not use?*

Exercise 4.3. *A stock has $\mu = 10\%$, $r = 2\%$, $\sigma = 16\%$. Compute the market price of risk. Write down the Girsanov density Z_T for horizon T. State the Q-dynamics of S. If μ rises to 18%, what changes in your answers and what does not?*

Exercise 4.4. *Write down the values of the digital call and the digital put with strike K. Verify their parity (their sum). Then evaluate both for the ATM data of Example 4.7 with the strike moved to $K = 110$. Use $\Phi(0.5766) = 0.7179$ and $\ln(100/110) = -0.09531$.*

Exercise 4.5. *Let $F_t = S_t e^{(r-q)(T^*-t)}$ be the futures price of §4.8. (a) Show directly from this formula and Theorem 4.14(a) that $dF_t = \sigma F_t \, d\widetilde{W}_t$ under Q. (b) Derive the put–call parity for options on futures, $C - P = e^{-r\tau}(F - K)$, by replication. (c) Conclude that the put in Example 4.16 is worth 7.57.*

Exercise 4.6. *Use put–call parity, $P = C - S + Ke^{-r\tau}$, and Theorem 4.11 to derive the five Greeks of the European put. Evaluate Δ_P, Γ_P, Θ_P, vega_P and ρ_P on the ATM numbers of Example 4.13.*

Exercise 4.7. *For the dividend-yield formula of Theorem 4.14(c), prove the identity $Se^{-q\tau}\varphi(d_1) = Ke^{-r\tau}\varphi(d_2)$ and deduce that $\partial C/\partial S = e^{-q\tau}\Phi(d_1)$.*

Exercise 4.8. *For the ATM data of Example 4.7, compute the two exercise probabilities $Q(S_T > K)$ and $Q^S(S_T > K)$. Explain their gap twice. First mechanically: compare d_1 with d_2. Then economically: what does counting value in shares do to the weight of the high-stock scenarios?*

Exercise 4.9. *Re-derive the Black–Scholes formula using only the share-measure decomposition $C = S\,Q^S(S_T > K) - Ke^{-r\tau}\,Q(S_T > K)$. Compute each probability from the terminal dynamics under its own measure. Completing the square is not allowed.*